CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 110b

D. G. MacMynowski — **Problem Set #1** — Issued: 4 Jan 10

Winter 2010 — Due: 11 Jan 10

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. In this problem, you will use the maximum principle to show that the shortest path between two points is a straight line. We model the problem by constructing a control system

$$\dot{x} = u,$$

where $x \in \mathbb{R}^2$ is the position in the plane and $u \in \mathbb{R}^2$ is the velocity vector along the curve. Suppose we wish to find a curve of minimal length connecting $x(0) = x_0$ and $x(1) = x_f$. To minimize the length, we minimize the integral of the velocity along the curve,

$$J = \int_0^1 \|\dot{x}\|\, dt = \int_0^1 \sqrt{\dot{x}^T \dot{x}}\, dt,$$

subject to to the initial and final state constraints. Use the maximum principle to show that the minimal length path is indeed a straight line at maximum velocity.

2. Consider the problem of moving a two-wheeled mobile robot (e.g., a Segway) from one position and orientation to another. The dynamics for the system is given by the nonlinear differential equation

$$\dot{x} = \cos\theta\, v, \qquad \dot{y} \ = \sin\theta\, v, \qquad \dot{\theta} = \omega,$$

where (x, y) is the position of the rear wheels, θ is the angle of the robot with respect to the x axis, v is the forward velocity of the robot and ω is spinning rate. We wish to choose an input (v, ω) that minimizes the time that it takes to move between two configurations (x_0, y_0, θ_0) and (x_f, y_f, θ_f), subject to input constraints $|v| \leq L$ and $|\omega| \leq M$.

Use the maximum principle to show that any optimal trajectory consists of segments in which the robot is traveling at maximum velocity in either the forward or reverse direction, and going either straight, hard left ($\omega = -M$) or hard right ($\omega = +M$).

Note: one of the cases is a bit tricky and cannot be completely proven with the tools we have learned so far. However, you should be able to show the other cases and verify that the tricky case is possible.

3. Derive Pontryagin's Maximum Principle for a discrete-time system with dynamics

$$x_{k+1} = f(x_k, u_k)$$

The cost functional is

$$J = \sum_{k=0}^{N-1} L(x_k, u_k) + V(x_N)$$

For simplicity, consider a fixed terminal time, with no constraint on the final state value (but with x_0 specified). Assume the control is constrained to lie in some set $u \in \Omega$.
Hint: Start by defining the Hamiltonian as

$$H(x, u, \lambda, k) = L(x_k, u_k) + \lambda_{k+1}^T f(x_k, u_k)$$