Constrained Function optimization

Given $F : \mathbb{R}^n \rightarrow \mathbb{R}$ and $G_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1 \ldots k$, then find $x^* \in \mathbb{R}^n$ such that $G_i(x^*) = 0 \forall i$ and $F(x^*) \geq F(x)$ for all $x$ satisfying $G_i(x) = 0 \forall i$.

- Then at optimal solution, gradient of $F(x)$ must be parallel to gradient of $G(x)$:
  \[
  \frac{\partial f}{\partial x} + \lambda \frac{\partial G}{\partial x} = 0
  \]
  - More generally, define:
    \[
    \bar{F} = F + \lambda^T G
    \]
  - Then a necessary condition is:
    \[
    \frac{\partial \bar{F}}{\partial x} (x^*) = 0
    \]
  - The Lagrange multipliers $\lambda$ are the sensitivity of the cost to a change in $G$
Optimal Control of Systems

Given a system:
\[ \dot{x} = f(x, u) \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^p \]
with \( x(0) = x_0 \). Then find
\[ u = \underset{u \in \Omega}{\text{argmin}} \left( \int_0^T L(x, u) dt + V(x(T), u(T)) \right) \]

- Easy to include additional constraints on control \( u \), and on state (along trajectory or at final time)
- Final time \( T \) may or may not be free (I’ll only derive for fixed \( T \))
- Define \( z = \begin{bmatrix} x \\ u \end{bmatrix} \), then this is a standard problem of minimizing \( J(z) \) subject to constraints \( G(z) = 0 \)

Solution approach

- Add Lagrange multiplier \( \lambda(t) \) for dynamic constraint
  - And additional multipliers for terminal constraints or state constraints
- Form augmented cost functional:
  \[ \tilde{J}(x, u, \lambda) = J(x, u) + \int_0^T \lambda^T (f(x, u) - \dot{x}) dt \\ - \int_0^T (L(x, u) + \lambda^T (f(x, u) - \dot{x})) dt + V(x(T)) \]
  \[ = \int_0^T (H(x, u) - \lambda^T \dot{x}) dt + V(x(T)) \]
  - where we introduce the Hamiltonian: \( H \triangleq L + \lambda^T f \)
  - A necessary condition for optimality is that \( \delta \tilde{J} \) vanishes for any perturbation in \( x, u, \) or \( \lambda \) about optimum:
    \[ x(t) = x^*(t) + \delta x(t) \]
    \[ u(t) = u^*(t) + \delta u(t) \]
    \[ \lambda(t) = \lambda^*(t) + \delta \lambda(t) \]

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Derivation...

\[ \delta J = J - J^* \]
\[ \simeq \int_0^T \left( \frac{\partial H}{\partial x} \delta x + \frac{\partial H}{\partial u} \delta u - \lambda^T \dot{\delta x} + \left( \frac{\partial H}{\partial \lambda} - \dot{x}^T \right) \delta \lambda \right) dt + \frac{\partial V}{\partial x} \delta x(T) \]

- Note that (integration by parts):
  \[ \int_0^T \lambda^T \delta x = - \int_0^T \lambda^T \delta x + \lambda^T(T) \delta x(T) - \lambda^T(0) \delta x(0) \]
- So:
  \[ \delta J = \int_0^T \left[ \left( \frac{\partial H}{\partial x} + \lambda^T \right) \delta x + \frac{\partial H}{\partial u} \delta u + \left( \frac{\partial H}{\partial \lambda} - \dot{x}^T \right) \delta \lambda \right] dt + \left( \frac{\partial V}{\partial x} - \lambda^T(T) \right) \delta x(T) + \lambda^T(0) \delta x(0) \]

Pontryagin's Maximum Principle

- If \((x^*, u^*)\) is optimal, then:
  \[ \dot{x} = \left( \frac{\partial H}{\partial \lambda} \right)^T \]
  \[ x(0) = x_0 \]
  \[ -\lambda = \left( \frac{\partial H}{\partial x} \right)^T \]
  \[ \lambda(T) = \left( \frac{\partial V}{\partial x \mid x=x(T)} \right)^T \]
  \[ H(x^*(t), u^*(t), \lambda^*(t)) \leq H(x^*(t), u, \lambda^*(t)) \quad \forall u \in \Omega \]

- If \(\Omega = \mathbb{R}^m\) and \(H\) differentiable then \(\partial H/\partial u = 0\)
- Can be more general and include terminal constraints
- Follows directly from:
  \[ \delta J = \int_0^T \left[ \left( \frac{\partial H}{\partial x} + \lambda^T \right) \delta x + \frac{\partial H}{\partial u} \delta u + \left( \frac{\partial H}{\partial \lambda} - \dot{x}^T \right) \delta \lambda \right] dt 
  + \left( \frac{\partial V}{\partial x} - \lambda^T(T) \right) \delta x(T) + \lambda^T(0) \delta x(0) \]
Interpretation of $\lambda$

$\dot{x} = \left( \frac{\partial H}{\partial \lambda} \right)^T x(0) = x_0$  \quad \leftarrow \quad \dot{x} = f(x, u)

$-\lambda = \left( \frac{\partial H}{\partial x} \right)^T \quad \lambda(T) = \left( \frac{\partial V}{\partial x} |_{x=T} \right)^T$

- Two-point boundary value problem: $\lambda$ is solved backwards in time
- $\lambda$ is the “co-state” (or adjoint variable)
- Recall that $H = L + \lambda^T f(x, u)$
  - If $L = 0$ (no integrated cost, only a final cost), then: $\dot{\lambda} = -\left( \frac{\partial f}{\partial x} \right)^T$
  - This is the tangent linear adjoint of the original system
    - I.e. the adjoint of the system linearized about the forward trajectory
    \[ \langle L^*(x), y \rangle = (x, L(y)) \]
- $\lambda(t)$ is the sensitivity of the cost to a perturbation in the state $x(t)$
  - In the integral as $\lambda(t) \delta \dot{x}$
  - Recall $\delta J = \ldots + \lambda(0) \delta x(0)$

\[ \delta J = \int \ldots \lambda^T \delta \dot{x} \ldots = \lambda^T(\tau) z \]

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2D flat plate (Won Tae Joe)

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General Nonlinear Optimization

• For a control input $u^k(t)$, compute the sensitivity:

\[
\begin{align*}
\dot{x} &= f(x, u^k(t)) \\
\lambda(t) &= -\lambda = (\partial H(x, u^k(t))/\partial x)^T \\
x(0) &\rightarrow x(T) \\
\lambda(T) &\rightarrow \text{Compute } \lambda(T) \text{ from } x(T)
\end{align*}
\]

• Given $\lambda(t)$, note gradient of cost is
  \[
  \frac{\delta J}{\delta u} = \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u}
  \]
  – In a linear problem, can take one step to the minimum
  – In nonlinear problem, take step in gradient direction
  – Do line-search along gradient direction, since forward iterations are typically cheap compared to adjoint
• With new guess for control input $u^{k+1}(t)$, iterate… until converged

Receding-horizon implementation

• The above iteration gives a (locally) optimal control trajectory that minimizes the cost over a horizon $[0,T]$ from state $x(0)=x_0$.
  – This may be sufficient to understand characteristics of system
• For real-time control:
  – Real horizon might be infinite, but finite computational power means can only compute over a finite horizon
  – In the presence of model uncertainty and disturbances, actual trajectory will not follow predicted trajectory
  – Approach: follow computed control trajectory over a (small) subset of the horizon $[0,t_k]$, measure new state, and recompute new optimum over interval $[t_k,T+t_k]$.
  – For some problems like Mars-landing, may have a compressing rather than receding horizon (i.e. fixed terminal time)
  – If terminal cost in formulation chosen to bound integrated cost over $[T,\infty]$ then cost is a Lyapunov function (hence stability guaranteed)

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**Linear system, Quadratic cost**

\[
\dot{x} = Ax + Bu \quad J = \frac{1}{2} \int_0^T \left( x^T Q x + u^T R u \right) dt
\]

- Apply PMP:
  \[
  \dot{\lambda} = -A^T \lambda + Q x \\
  Ru = -B^T \lambda 
  \]

  \(B^T\) selects the part of the state that is influenced by \(u\), so \(B^T \lambda\) is sensitivity of aug. state cost to \(u\)

- Guess that \(\lambda(t) = P(t)x(t)\):
  \[
  -\dot{P} = PA + A^T P + Q - PBR^{-1}B^T P \\
  u = -R^{-1}B^T P x 
  \]

  \(-\dot{P}\) has an interpretation as the "cost to go"

  Often see the infinite-horizon solution where \(dP/dt = 0\)

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Examples

- Note adjoint solution alone is useful
  - Weather forecasting, to understand what measurements are most useful to make, and to propagate uncertainty (i.e. sensitivity to initial condition errors)
  - Understanding what perturbations are most likely to lead to an El Niño event
- Theory
  - Use PMP to prove that bang-bang control is time-optimal, for example
- For control
  - Real-time implementation of full non-linear limited to relatively simple systems (e.g. chemical plants, Mars entry/descent/landing (EDL),…)
  - Used to understand limits of performance, and characteristics…
    • 2D separation over airfoil (Won Tae)
    • El Niño control (me)
    • Turbulent boundary layers (e.g. Bewley et al)
    • Jet noise (Freund, Colonius)

El Niño Dynamics (chaotic system)

- Forward model is Fortran-77 legacy code (Cane & Zebiak)
- Adjoint model obtained from automatic differentiation (Adifor)
- Compare linear feedback to optimal control

Fig. 2. SST anomaly predicted by model at the peak of an El Niño event (at ~1997 in the uncontrolled time-history in Fig. 4(b)). The Niño-3 region is shown boxed. For control, solar insolation is dynamically varied over the eastern half of the Niño-3 region.

Fig. 3. Performance vs. control effort, comparing SISO linear feedback with the optimal nonlinear control, for a 15-yr simulation.
Summary

- Necessary conditions to solve a general non-linear optimal control problem are straightforward to write down
  - Only characterizes extrema; problem is not in general convex!
- The solution involves the adjoint
  - In general, a Lagrange multiplier is the sensitivity of the cost to a change in the constraint
  - Specifically for this dynamic constraint, the Lagrange multiplier is the sensitivity of the cost to a change in the state at each time
    \[ \delta \dot{x} = z \delta(t - \tau) \]
    \[ \Rightarrow \delta J = \int \cdots \lambda^T \delta \dot{x} \cdots = \lambda^T(\tau) z \]
    \[ x(\tau^+) = x(\tau^-) + z \]
- The solution in general is iterative
  - Requires solving 2PBVP
- For linear system, quadratic cost, then a closed-form solution exists
  - Called LQR (Linear Quadratic Regulator)