Discrete-Time Covariance Propagation

$x_{k+1} = A x_k + F v_k$ where v is white noise, zero-mean Gaussian

$$E\{v(\lambda) v^T(\xi)\} = Q_v\,\delta(\lambda - \xi)$$
$$= \begin{cases} Q_v & \lambda = \xi \\ 0 & \lambda \neq \xi \end{cases}$$

Then

1) $E\{x_{k+1}\} = A^{k+1} E\{x_0\}$

2) Define $P_k = E\{x_k x_k^T\}$, with $P_0 = E\{x_0 x_0^T\}$

Then
$$P_{k+1} = E\{(A x_k + F v_k)(A x_k + F v_k)^T\}$$
$$= E\{A x_k x_k^T A^T\} + E\{A x_k v_k^T F^T\}$$
$$+ E\{F v_k x_k^T A^T\} + E\{F v_k v_k^T F^T\}$$

$$\boxed{P_{k+1} = A P_k A^T + F Q_v F^T}$$

(Note x_k depends on $v_{k-1}, v_{k-2}, \ldots$ but v_k is independent of v_{k-1} since white $\Rightarrow E\{x_k v_k^T\} = E\{x_k\} E\{v_k\}^T$
$= 0$

3)
In steady-state, $P_{k+1} = P_k$ (if stable)
and
$$\boxed{P = APA^T + FQF^T}$$

4) Note: $P_{k+1} = \underbrace{A P_k A^T}_{\text{decrease (if stable) due to propagation (decay)}} + \underbrace{F Q_v F^T}_{\text{increase due to additional noise}}$

5) If $y_k = C x_k$
$E\{y_k y_k^T\} = C P C^T$

Discrete-Time Kalman Filter

Given $x_{k+1} = Ax_k + Bu_k + Fv_k$

$$y_k = Cx_k + w_k$$

where v, w are Gaussian, white noise

$$E\{v_k\} = 0$$

$$E\{v_k v_j^T\} = Q\delta(k-j) = \begin{cases} Q & k=j \\ 0 & k \neq j \end{cases}$$

$$E\{w_k\} = 0$$

$$E\{w_k w_j^T\} = R\delta(k-j) = \begin{cases} R & k=j \\ 0 & k \neq j \end{cases}$$

$$E\{v_k w_k^T\} = 0 \quad \text{(uncorrelated)}$$

& $E\{x_0\} = x_0, \quad E\{x_0 x_0^T\} = P_0$

Then Find $\hat{x}_k$ to minimize $E\{(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T\}$ given y_j, $j = 0, 1, \ldots k$

Solution:

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + L_k(y - C\hat{x}_k)$$

where

$$L_k = AP_k C^T (R + CPC^T)^{-1}$$

& $P_{k+1} = APA^T + Q - AP_k C^T (R + CPC^T)^{-1} CPA^T$

Proof

Error dynamics satisfy

$$e_{k+1} = (A - LC)e_k - L_k w_k + Fv_k$$

From before, covariance $P_k = E\{e_k e_k^T\}$ satisfies

$$P_{k+1} = (A-LC)P_k(A-LC)^T + FQF^T + LRL^T$$

$$= AP_kA^T + FQF^T - LCP_kA^T - AP_kC^TL^T + L(R+CP_kC^T)L^T$$

minimizing $Tr(P_{k+1})$ gives

$$\frac{\partial}{\partial L} = \left(-AP_kC^T + L(R+CPC^T)\right) + (\quad)^T = 0$$

$$\Rightarrow L = APC^T(R+CPC^T)^{-1}$$

& substituting gives

$$P_{k+1} = AP_kA^T + FQF^T - AP_kC^T(R+CP_kC^T)^{-1}CP_kA^T$$

In steady-state, $P_{k+1} = P_k$ &

if (A, C) observable, $(A, F\sqrt{Q})$ controllable,

$\exists\, P > 0$ satisfying

$$P = APA^T + FQF^T - APC^T(R+CPC^T)^{-1}CPA^T$$

<u>Predictor - corrector form</u>

The update equations can be separated into two steps:

0) initialize: $\hat{x}_{0|-1} = E\{x_0\}$, $P_{0|-1} = E\{x_0x_0^T\}$ $\Big\}$ Prior information

Then $\forall k$:

1) Correction $\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k(y_k - C\hat{x}_{k|k-1})$

write first

2) Prediction $\hat{x}_{k+1|k} = A\hat{x}_{k|k} + Bu_k$

given $y_0, \dots y_k$

Update covariances as

1) correction

$$P(k|k) = P(k|k-1) - P(k|k-1)\,C^T\left(C P(k|k-1)\, C^T + R\right)^{-1} C P_{k|k-1}$$

2) Prediction

$$P(k+1|k) = A P(k|k) A^T + F Q F^T$$

Remarks

1. In steady-state, covariance will increase during prediction step & decrease in correction step.

2. Main advantage of this form:
   - easy to incorporate different information at different rates
   - easy to know what to do with missing information
   - can update control based on best estimate at the time (e.g. might not yet have y_k, but can still estimate $\hat{x}_{k|k-1}$)

Discrete-time Kalman Filter
Predictor-Corrector form
D. MacMynowski
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The discrete-time Kalman filter can be written as two steps; prediction (what is the estimate and covariance in the future, given the current values) and correction (updating the estimate and covariance at the current time, given new measurement information). This Predictor-Corrector form is useful for incorporating different types of sensor information at different rates, understanding what to do if a measurement is missing at a given time, etc.

For notation, I use the subscript $(\cdot)_{k|j}$ to indicate the value of $(\cdot)$ at time step k, given all information up to and including that at time-step j. The algorithm proceeds as follows:

1. Initialize the estimate and covariance at $k = 0$ with the best information available:

$$\begin{aligned}\hat{x}_{0|-1} &= E\left\{x_0\right\} \\ P_{0|-1} &= E\left\{x_0 x_0^T\right\}\end{aligned}$$

2. *Correction:* Given a new measurement at time-step $k \geq 0$:

$$\begin{aligned}\hat{x}_{k|k} &= \hat{x}_{k|k-1} + L_k(y - C\hat{x}_{k|k-1}) \\ P_{k|k} &= P_{k|k-1} - P_{k|k-1}C^T\left(CP_{k|k-1}C^T + R\right)^{-1}CP_{k|k-1}\end{aligned}$$

   where

$$L = P_{k|k-1}C^T\left(CP_{k|k-1}C^T + R\right)^{-1} = P_{k|k}C^T R^{-1}$$

   Note that the expression for $P_{k|k}$ can be re-written using the matrix inversion lemma in "information" form as

$$P_{k|k}^{-1} = P_{k|k-1}^{-1} + C^T R^{-1} C$$

   This is not a computationally efficient way to implement the filter, but is useful both because it allows initializing the covariance from a state of zero information ($P_{0|-1}^{-1} = 0$), and because it illustrates clearly that simultaneous measurements from multiple sensors can be incorporated into the estimate sequentially in any order; the information is increased by the sum of the information from each measurement.

3. *Prediction:* Given the best estimate at one time-step, the best estimate at a future time is given by

$$\begin{aligned}\hat{x}_{k+1|k} &= A\hat{x}_{k|k} + Bu_k \\ P_{k+1|k} &= AP_{k|k}A^T + FRF^T\end{aligned}$$

Note that in the previously derived form of the discrete-time Kalman filter equations (not separated into prediction and correction steps), then the estimate $\hat{x}_{k+1}$ is $\hat{x}_{k+1|k}$, and similarly with the covariance. The optimal filter gain in the previous form is related to that used in the predictor-corrector equations by a factor of A.