Continuous-Time Covariance Propagation

Consider the system

$$\dot{x} = Ax + Fv \qquad x \in \mathbb{R}^n \quad v \in \mathbb{R}^m, \; y \in \mathbb{R}^p$$
$$y = Cx$$

where v is Gaussian white noise,

$$E\{v(\tau)v^T(\eta)\} = Q\,\delta(t-\eta)$$

Covariance of $x(t)$ is $P(t) = E\{x(t)x^T(t)\}$
$\left(\& \quad P(0) = E\{x(0)x(0)^T\} = P_0\right)$

Then
$$P(t) = E\left\{\left(e^{At}x_0 + \int_0^t e^{A(t-\tau)}Fv(\tau)\,d\tau\right) \times \left(e^{At}x_0 + \int_0^t e^{A(t-\eta)}Fv(\eta)\,d\eta\right)^T\right\}$$

$$= E\left\{e^{At}x_0x_0^Te^{A^Tt}\right\}$$
$$+ E\left\{\int_0^t e^{At}x_0v(\tau)^TF^Te^{A^T(t-\tau)}\,d\tau\right\} + (\ldots)^T$$
$$+ E\left\{\int_0^t\int_0^t e^{A(t-\tau)}Fv(\tau)v^T(\eta)F^Te^{A^T(t-\eta)}\,d\tau\,d\eta\right\}$$

But $E\{x_0v(\tau)^T\} = 0$, $E\{v(\tau)v^T(\eta)\} = Q\delta(t-\eta)$

$$\Rightarrow \quad P(t) = e^{At}P_0e^{A^Tt} + \int_0^t e^{A(t-\tau)}FQF^Te^{A^T(t-\tau)}\,d\tau$$

So
$$\boxed{\dot{P}(t) = \underbrace{AP + PA^T}_{\text{change from state evolution}} + \underbrace{FQF^T}_{\text{increment from noise}}}$$

&
$E\{yy^T\}$
$= CPC^T$

& $P(t, t+\tau) = P(t)e^{-A\tau}$

In steady state, if stable (A Hurwitz)

Then

$$\text{(1)} \quad \boxed{AP + PA^T + FQF^T = 0}$$

Lyapunov Eq'n

Recall cost to go for quadratic cost

$$\int_t^\infty x^T Q_c x = x(t)^T P_c x(t)$$

where

$$\text{(2)} \quad \boxed{P_c A + A^T P + Q_c = 0}$$

P_c satisfying (2) The observability Gramian for $(A, \sqrt{Q_c})$;

If $\exists P_c > 0$ (& A stable) Then system is observable

If $(A, \sqrt{Q_c})$ observable Then $V = x^T P_c x$ is a Lyap Fn

P satisfying (1) is The controllability Gramian for $(A, F\sqrt{Q})$

If A stable & $\exists P > 0$ Then system is ~~controllable~~ reachable

Gramians are useful in showing how ctrllable & obsv. a system is, and also relevant in model reduction (matching input/output behaviour with fewer states by discarding almost unobservable or uncontrollable states)

Kalman Filter

$$\dot{x} = Ax + Bu + Fv \qquad \qquad , \quad v, w \text{ Gaussian, zero-mean,}$$
$$y = Cx + w \qquad \qquad \text{white, covariance } Q \;\&\; R$$

Minimize $\quad E\left\{(x-\hat{x})(x-\hat{x})^T\right\}$ given y

OR (equivalent), $\quad \hat{x}(t) = E\left\{x(t) \,\middle|\, y(\tau),\ \tau \le t\right\}$

(expectation of the random process same (max likelihood) as minimum variance)

... generalization of least squares fitting to dynamics

The optimal estimate is a linear observer

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

where L is obtained from a Riccati ODE

Derivation:

Error dynamics are

$$\dot{e} = (A - LC)e + \xi \qquad \text{where} \quad \xi = Fv - Lw$$
$$R_\xi = FQF^T + LRL^T$$

The covariance matrix $P_e = P$

satisfies

$$\dot{P} = (A - LC)P + P(A - LC)^T + FQF^T + LRL^T$$

To make P as small as possible, $\frac{\partial(RHS)}{\partial L} = 0$ (See Friedland 9.4)

$$\Rightarrow \quad -PC^T + LR = 0, \quad \boxed{L = PC^TR^{-1}}$$

$$\Rightarrow \quad \boxed{\dot{P} = AP + PA^T + FQF^T - PC^TR^{-1}CP} \quad \text{and } P > 0$$

Remarks and properties

1. Kalman filter has the form of a recursive filter: given $P(t) = E\{e(t)\, e^T(t)\}$ at time t, can compute how estimate & covariance changes. Don't need to keep track of old values of output $y(t)$.

2. Kalman filter gives estimate $\hat{x}(t)$ and covariance $P_e(t)$ $\Rightarrow$ you can see how well error is converging.

3. If noise is stationary (Q, R constant) and if $\dot{P}$ is stable, then observer gain is constant

$$L = PC^T R^{-1} \qquad AP + PA^T - PC^T R^{-1} CP + FQF^T \leftarrow \text{Algebraic Riccati Eq}$$

This is the problem solved by lqe command in MATLAB.

4. Kalman filter extracts maximum possible information about output data:

$$r = y - C\hat{x} \;=\; \text{residual or } \underline{\text{innovations process}}$$

Can show that for the Kalman filter, correlation matrix is

$$R_r(t, s) = W(t)\, \delta(t-s) \;\Rightarrow\; \text{white noise}$$

So output error has no remaining dynamic information content. (See Friedland Sec 11.5 for complete calculation)