Chapter 2

Model Predictive Control

In traditional receding- or compressing-horizon implementations of MPC (Model Predictive Control), control inputs are computed online by solving an FHC (Finite-Horizon optimal Control problem) over a finite time horizon, subject to state and control constraints, and with the current state of the system as the initial state. The control is then applied to the system in a feedforward (i.e., open-loop) manner over a specified time interval, followed by an update to the current state and a re-solve (re-computation) of the FHC over a compressed or receded time horizon. The re-solve provides an updated feedforward input, which is then applied to the system and the cycle repeats.

The intent of this chapter is to define the baseline MPC method that is used for comparison with the contributions in this thesis. The MPC formulation presented herein is based upon those by Chen and Allgöwer [13], Mayne *et al.* [31], Jadbabaie [18] and Primbs [37] where a CLF (Control Lyapunov Function [46, 47, 16]) is imposed on the terminal state as part of the cost function. The method additionally uses a terminal state constraint, as well as trajectory state and control constraints. Refer back to Section 1.1 of Chapter 1 for a review of the other relevant literature and contributions to MPC. Proofs for the lemmas and theorem in this chapter are provided in Appendix A in lieu of external references because future chapters build upon and extend them.

2.1 System Description and Control Objective

Consider the following nonlinear system as the *nominal* system for application of MPC:

$$\dot{z} = F(z, u_o, t), \tag{2.1}$$

with nominal state $z \in \mathbb{R}^n$, control input $u_o \in \mathbb{R}^m$, and $F(\cdot) \in \mathbb{R}^n$ is perfectly known (i.e., there are no parametric uncertainties or unknown disturbances). Without a loss of generality, the origin is considered an equilibrium point for system (2.1), $F(0,0,t) = 0$; any non-zero equilibrium point $F(\bar{z}, \bar{u}_o, t) = 0$ can be shifted to the origin.

The control objective is to obtain control input u_o that, when applied to nominal system (2.1), renders the origin ($z = 0$) asymptotically stable, with a region of attraction $\mathcal{R}_n \subseteq \mathbf{Z}_n$, such that

$$z(t) \in \mathbf{Z}_n \text{ and } u_o(t) \in \mathbf{U}_o, \ \forall\, t \geq t_0, \tag{2.2}$$

when $z(t_0) \in \mathcal{R}_n$. Sets $\mathbf{Z}_n \subseteq \mathbb{R}^n$ and $\mathbf{U}_o \subseteq \mathbb{R}^m$ define nominal state and control constraints, respectively: $\mathbf{Z}_n$ is connected and contains the origin in its interior; $\mathbf{U}_o$ is compact and contains the origin in its interior. The set $\mathcal{R}_n$ will be defined based on the architecture of the MPC algorithm.

2.2 Architecture of MPC Algorithm

The following FHC is typical of the type of constrained optimization solved in an MPC framework. The FHC finds a control input u_o that minimizes an objective function over a finite time horizon ($T \geq 0$), subject to the dynamics of nominal system (2.1), the imposed state and control constraints, and a terminal state constraint.

FHC (for MPC)

Find $J^* = \min_{u_o} J(z, u_o;\ t_i, T, z(t_i))$ where

$$J(z, u_o;\ t_i, T, z(t_i)) = \int_{t_i}^{t_i+T} h(z(\tau), u_o(\tau))d\tau + V(z(t_i + T))$$

subject to

$$\left.\begin{aligned} &\dot{z} = F(z, u_o, t), \\ &z(t) \in \mathbf{Z}_n, \\ &u_o(t) \in \mathbf{U}_o, \end{aligned}\right\} \forall\, t \in [t_i,\, t_i + T]$$

$$z(t_i + T) \in \mathbf{\Omega}_o,$$

where $z(t_i)$ is the nominal system (2.1) state at initial time t_i.

The set $\mathcal{R}_n$ will be defined based on feasibility of the FHC:

$$\mathcal{R}_n = \{\xi \in \mathbf{Z}_n : \text{ FHC is feasible with } z(t_i) = \xi\}. \tag{2.3}$$

The following conditions on the FHC and the nominal system (2.1) are instrumental for proving asymptotic stability (e.g., [13, 18]) of the MPC algorithm:

Condition 2.1. Function $h(\cdot)$ is positive definite [21], satisfying

$$h(z, u_o) \geq a||z||^p + b||u_o||^r, \quad \forall z, u_o, \tag{2.4}$$

with $p \geq 1$, $r \geq 0$, a and b both positive constants, and $h(0,0) = 0$. ◇

Condition 2.2. Function $V(z)$ is positive definite ($V(z) > 0, \forall z \neq 0,$ with $V(0) = 0$), and there exists a control law $u_o = \mathcal{L}(z)$ such that V defines a Control Lyapunov Function for (2.1) satisfying

$$\nabla V(z) F(z, \mathcal{L}(z), t) + h(z, \mathcal{L}(z)) \leq 0, \quad \forall z \in \mathbf{\Omega}_o, \tag{2.5}$$

where $\mathbf{\Omega}_o \subset \mathbf{Z}_n$ is compact, convex, contains the origin in its interior, and is invariant for dynamics (2.1) under application of control policy $\mathcal{L}(z)$. Additionally, $\mathcal{L}(z) \in \mathbf{U}_o$, $\forall z \in \mathbf{\Omega}_o$. ◇

Condition 2.3. There exists closed ball† $\mathbf{B}_R$ centered at the origin such that set $\mathbf{\Omega}_o$ satisfies

$$\mathbf{B}_R \subseteq \mathbf{\Omega}_o. \tag{2.6}$$

◇

Note that invariance of $\mathbf{\Omega}_o$ under application of control law $\mathcal{L}(z)$ implies that if $z(t_0) \in \mathbf{\Omega}_o$ for some t_0, then $z(t) \in \mathbf{\Omega}_o$, $\forall t \geq t_0$ [21]. Additionally, Conditions 2.1 and 2.2 imply that cost function J in the FHC is also positive definitive ($J(z, u_o) > 0, \forall z, \forall u_o,$ with $J(0,0) = 0$).

The following algorithm describes the MPC approach. Note, the superscript k on $u_o^k(t)$ and $z^k(t)$ in the below MPC algorithm denotes the feedforward input and the resulting nominal trajectory, respectively, associated with a re-solve at time t_k.

---

† $\mathbf{B}_\rho \triangleq \{z : \|z\| \leq \rho,\ \rho > 0\}$.

MPC Algorithm

Begin at $k = 0$ with $z(t_0) \in \mathcal{R}_n$ and iterate the following steps over re-solve times t_k for $k \in \mathbb{Z}^+$:

1. Measure state $z(t_k)$ of nominal system (2.1) and solve the FHC at time $t_i = t_k$ with $z(t_i) = z(t_k)$ and $T = T_k$ to obtain $u_o^k(t)$ on $t \in [t_k, t_k + T_k]$.

2. Monitor $z(t)$ while applying $u_o(t) = u_o^k(t)$ to nominal system (2.1) on $t \in [t_k, t_{k+1}]$, with $z(t) = z^k(t)$.

3. Check the following over $t \in [t_k, t_{k+1}]$:
   if $z(\tilde{t}) \in \mathbf{\Omega}_o$ for some $\tilde{t} \geq t_0$, then set $u_o(t) = \mathcal{L}(z), \forall t \geq \tilde{t}$ and stop iteration.

Lemma 2.1 (Re-solvability of the FHC)**.** *Suppose that the FHC is feasible at t_0 with horizon T_0, and let t_k for $k \in \mathbb{Z}^+$ be the times that a solution of the FHC is re-solved. Then, the feasibility of the FHC is guaranteed at t_k with $T_k \geq T_{k-1} - \delta_k$, $\forall k \in \mathbb{Z}^+$, $\delta_k = t_k - t_{k-1}$, $0 \leq \delta_k < T_{k-1}$ provided Condition 2.2 holds.* ◇

Proof. See Appendix A.1 for a proof of Lemma 2.1. □

For proving stability of the MPC algorithm, a sequence of monotonically increasing re-solve times is needed:

Definition 2.1 (Re-Solve Times)**.** Let t_k ($k \in \mathbb{Z}^+$) be *re-solve times* for the FHC satisfying $\inf_k \delta_k \geq \epsilon$ for some $\epsilon > 0$ where $\delta_k = t_k - t_{k-1}$. ◇

Lemma 2.2 (Shrinking Optimal Cost with Compressing or Receding Horizon)**.** *Suppose the FHC is feasible at some re-solve time t_{k-1} and T_{k-1} with optimal cost J_{k-1}^*, and Conditions 2.1, 2.2, and 2.3 hold. Then, the FHC is feasible at re-solve time t_k with $T_k \in [T_{k-1} - \delta_k, T_{k-1}]$ (in fact any $T_k \geq t_{k-1} - \delta_k$), and if $z^{k-1}(t_{k-1}) \notin \mathbf{\Omega}_o$ and $z^{k-1}(t_k) \notin \mathbf{\Omega}_o$, then the optimal cost satisfies*

$$J_k^* - J_{k-1}^* \leq -\beta, \quad \textit{for some} \quad \beta > 0. \tag{2.7}$$

◇

Proof. See Appendix A.2 for a proof of Lemma 2.2. □

While each solution of the FHC provides a feedforward input u_o to drive the nominal system (2.1) toward the origin, the ability to re-solve the FHC and thus update the feedforward input u_o based on the current state provides closed-loop feedback. The following theorem establishes closed-loop asymptotic stability and finite-time convergence of the MPC algorithm:

Theorem 2.1 (Closed-Loop Asymptotic Stability of MPC)**.** *Consider system (2.1) for z and control input u_o described by the MPC algorithm. If Conditions 2.1, 2.2, and 2.3 are satisfied, then the origin ($z = 0$) of the resulting closed-loop system is asymptotically stable with region of attraction $\mathcal{R}_n$.* ◇

Proof. See Appendix A.3 for a proof of Theorem 2.1. □

2.3 Implementation and Limitations

Practical implementation of the MPC algorithm can be difficult due to online computational capability, measurement and computation delay, parametric uncertainty, and unknown exogenous disturbances. These sources of error can lead to difficulty in maintaining feasibility, and thus re-solvability, of the FHC. Rather than providing a specific example to demonstrate the effect of uncertainty, a graphic illustration of the MPC algorithm applied to a constrained nominal system will be contrasted. A specific example demonstrating these issues will be given in the next chapter on Robust MPC, where a contrast is made between the robust method and the baseline MPC method of this chapter.

For applications of the MPC algorithm, the nominal system in (2.1) serves as a model for the actual system

$$\dot{x} = f(x, u, t), \tag{2.8}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. The actual system contains uncertainty in either the parameters or from unknown, exogenous disturbances, and thus $F(\cdot)$ in (2.1) is a known, idealized model of $f(\cdot)$ in (2.8).

The asymptotic stability guarantees for the MPC algorithm require the nominal trajectory to remain on the computed trajectory under application of feedforward policy u_o between re-solve times. This provides an initial, feasible nominal state for $z(t)$ at subsequent re-solves, as depicted in the left-side sketch in Figure 2.1. However, if there is error

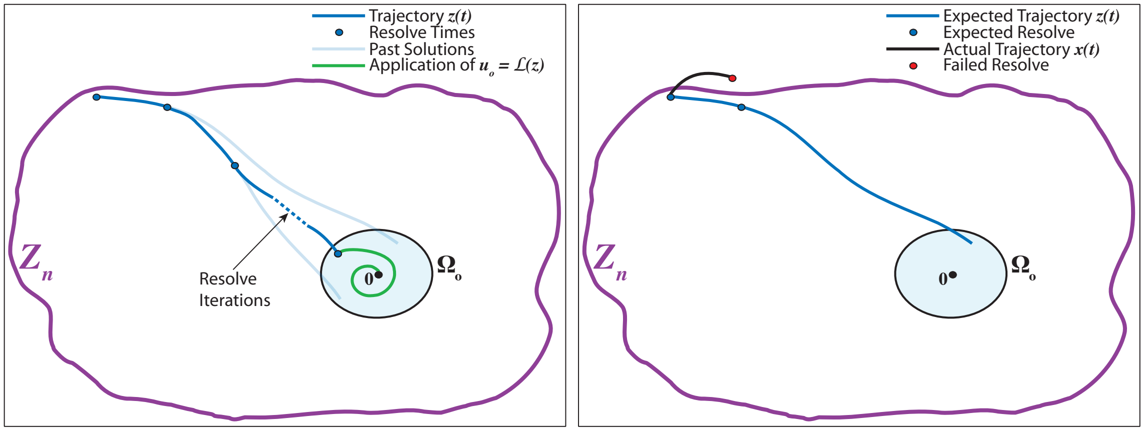

Figure 2.1: Expected MPC trajectories (left) and system uncertainty causing an infeasible state at re-solve (right).

in the nominal model, and $F(\cdot) \neq f(\cdot)$, then the actual trajectory $x(t)$ will not remain on the expected trajectory $z(t)$, thus providing no guarantee of an initial, feasible state for the FHC at the re-solve time. In fact, the actual state $x(t)$ can even violate the nominal system state constraints $\mathbf{Z}_n$, which also renders the MPC algorithm infeasible at a re-solve, as depicted in the right-side sketch of Figure 2.1.

Appendix A

Relevant Proofs for Continuous-Time Algorithms

A.1 Proof of Lemma 2.1: Re-Solvability of the FHC

Proof. (by induction) Suppose at t_{k-1} the FHC is feasible with T_{k-1} and provides $u_o^{k-1}(t)$ for $t \in [t_{k-1}, t_{k-1} + T_{k-1}]$. Let $z^{k-1}(t)$ be the trajectory of nominal system (2.1) for $t \in [t_{k-1}, t_{k-1} + T_{k-1}]$ corresponding to control input $u_o^k(t)$.

Let $t_k = t_{k-1} + \delta_k$ and re-solve the FHC with T_k. Since $F(\cdot)$ in nominal system (2.1) is known perfectly, the resulting nominal state $z(t)$ is exactly $z^{k-1}(t)$ from application of $u_o^{k-1}(t)$ for $t \in [t_{k-1}, t_k]$; thus, state $z^k(t_k) = z^{k-1}(t_k)$ remains the initial state of a feasible trajectory. Then, the following control input provides a feasible solution to the FHC re-solve:

$$u_o^k(t) = \begin{cases} u_o^{k-1}(t), & t \in [t_k, t_{k-1} + T_{k-1}] \\ \mathcal{L}(z(t)), & t \in [t_{k-1} + T_{k-1},\, t_k + T_k]\,, \end{cases} \tag{A.1}$$

where $T_k \geq T_{k-1} - \delta_k$, and thus $t_k + T_k \geq t_{k-1} + T_{k-1}$. Input $u_o^k(t) = u_o^{k-1}(t) \in \mathbf{U}_o$ provides $z^k(t) = z^{k-1}(t) \in \mathbf{Z}_n$ for $t \in [t_k, t_{k-1} + T_{k-1}]$. Since $z^k(t_{k-1} + T_{k-1}) = z^{k-1}(t_{k-1} + T_{k-1}) \in \mathbf{\Omega}_o$ and Condition 2.2 holds, then $\mathbf{\Omega}_o$ is invariant and $z(t) \in \mathbf{\Omega}_o$ with $u^k(t) = \mathcal{L}(z(t)) \in \mathbf{U}_o, \forall t \in [t_{k-1} + T_{k-1},\, t_k + T_k]$.

Thus, control policy (A.1) provides a feasible solution to the FHC re-solve at t_k for any $T_k \geq T_{k-1} - \delta_k$ once the FHC is feasible at t_{k-1} with T_{k-1}. □

A.2 Proof of Lemma 2.2: Shrinking Optimal Cost with Receding Horizon

Proof. Since the FHC is feasible at t_{k-1} with T_{k-1} and $u_o^{k-1}(t)$ provides the optimal cost J^*_{k-1}, then $u_o^{k-1}(t)$ can also be used to provide a feasible solution for the FHC at t_k with $T_k \in [T_{k-1} - \delta_k, T_{k-1}]$ (or any $T_k \geq t_{k-1} - \delta_k$) by using $u_o^k(t)$ in (A.1) from the proof of Lemma 2.1. So, $z^k(t) = z^{k-1}(t)$ is a feasible trajectory for $t \in [t_k, t_{k-1} + T_{k-1}]$.

From Definition 2.1 for the computation times, $t_k - t_{k-1} = \delta_k \geq \epsilon > 0$. Thus, $t_k + T_k \geq t_{k-1} + T_{k-1}$ when $T_k \geq T_{k-1} - \delta_k$. Note, $T_k \in [T_{k-1} - \delta_k, T_{k-1}]$ covers standard compressing- and receding-horizon implementations of MPC.

The cost at t_k with control input (A.1) can be written as

$$J_k = \int_{t_k}^{t_{k-1}+T_{k-1}} h\left(z^{k-1}(\tau), u_o^{k-1}(\tau)\right) d\tau \; + \int_{t_{k-1}+T_{k-1}}^{t_k+T_k} h\left(z^k(\tau), u_o^k(\tau)\right) d\tau \; + \; V\left(z^k(t_k + T_{k-1})\right).$$

Note, the second integral (with limits $t_{k-1} + T_{k-1}$ to $t_k + T_k$) is over a non-negative interval since $t_k + T_k \geq t_{k-1} + T_{k-1}$ as shown. At t_{k-1}, the optimal cost can be written as

$$J^*_{k-1} = \int_{t_{k-1}}^{t_k} h\left(z^{k-1}(\tau), u_o^{k-1}(\tau)\right) d\tau + \int_{t_k}^{t_{k-1}+T_{k-1}} h\left(z^{k-1}(\tau), u_o^{k-1}(\tau)\right) d\tau + V\left(z^{k-1}(t_{k-1} + T_{k-1})\right),$$

and thus $J_k - J^*_{k-1} =$

$$\int_{t_{k-1}+T_{k-1}}^{t_k+T_k} h\left(z^k(\tau), u_o^k(\tau)\right) d\tau - \int_{t_{k-1}}^{t_k} h\left(z^{k-1}(\tau), u_o^{k-1}(\tau)\right) d\tau + V\left(z^k(t_k + T_k)\right) - V\Big(\underbrace{z^{k-1}(t_{k-1} + T_{k-1})}_{= z^k(t_{k-1}+T_{k-1})}\Big).$$

Condition 2.2 implies the following with $u_o^k(t) = \mathcal{L}(z^k(t))$ on $t \in [t_{k-1} + T_{k-1}, t_k + T_k]$:

$$\int_{t_{k-1}+T_{k-1}}^{t_k+T_k} \dot{V}\left(z^k(\tau)\right) d\tau \; + \int_{t_{k-1}+T_{k-1}}^{t_k+T_k} h\left(z^k(\tau), u_o^k(\tau)\right) d\tau \leq 0$$

and

$$V\left(z^k(t_k + T_k)\right) - V\left(z^k(t_{k-1} + T_{k-1})\right) \; + \int_{t_{k-1}+T_{k-1}}^{t_k+T_k} h\left(z^k(\tau), u_o^k(\tau)\right) d\tau \leq 0.$$

This implies that

$$J_k - J^*_{k-1} \leq -\int\limits_{t_{k-1}}^{t_k} h\left(z^{k-1}(\tau), u_o^{k-1}(\tau)\right) d\tau. \tag{A.2}$$

Given Conditions 2.1 and 2.3, if $||z|| \geq R$, then $h(z, u_o) = \rho > 0$. Since $t_k - t_{k-1} = \delta_k \geq \epsilon > 0$ by Definition 2.1 for the computation times, then

$$\int\limits_{t_{k-1}}^{t_k} h(z_{k-1}(\tau), u_{o,k-1}(\tau)) d\tau \geq \underbrace{\rho\epsilon}_{\beta} > 0, \tag{A.3}$$

where $\beta > 0$ is independent of k.

Combining inequalities (A.2) and (A.3) shows that $J_k - J^*_{k-1} \leq -\beta < 0$, and since $J^*_k \leq J_k$, then

$$J^*_k - J^*_{k-1} \leq -\beta < 0.$$

□

A.3 Proof of Theorem 2.1: Closed-Loop Asymptotic Stability of MPC

Proof. Given the MPC algorithm and $z(t_0) \in \mathcal{R}_n$ such that the FHC is feasible with some $T = T_0$, suppose there exists $k \in \mathbb{Z}^+$ such that $z^{k-1}(t_{k-1}) \notin \mathbf{\Omega}_o$ and $z^{k-1}(t_k) \notin \mathbf{\Omega}_o$. Then, $z^{k-1}(t) \notin \mathbf{\Omega}_o$ for $t \in [t_{k-1}, t_k]$, and (2.7) holds. Consequently, if the nominal trajectory z does not enter $\mathbf{\Omega}_o$ in finite time, then there exists $k \in \mathbb{Z}^+$ such that $J^*_k < 0$, which is a contradiction. This, together with Condition 2.2, imply the existence of finite time $\tilde{t} \geq t_0$ such that $z(t) \in \mathbf{\Omega}_o$, $\forall t \geq \tilde{t}$.

Application of Step 3 in the MPC algorithm, $u_o = \mathcal{L}(z)$ for $t \geq \tilde{t}$, and use of Condition 2.2 imply

$$\lim_{t\to\infty} \|z(t)\| = 0$$

since V is a Control Lyapunov Function for nominal system (3.2) with $\dot{V}(z) < 0, \forall z$, except $\dot{V}(0) = 0$. Therefore, the closed-loop nominal system (2.1) converges asymptotically to the origin, $\forall z(t_0) \in \mathcal{R}_n$, with control input u_o given by the MPC algorithm. □