CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 101

D. G. MacMartin | **Problem Set #4** | Issued: 20 Oct 14
Fall 2014 | | Due: 29 Oct 14

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. Consider the normalized, linearized inverted pendulum which is described by

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u = Ax + Bu, \qquad y = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Cx$$

   (a) Compute the reachability matrix for the system and show that the (linearized) system is reachable.

   (b) Determine a state feedback and reference gain $u = -Kx + k_r r$ that gives a closed loop system with unit static gain (steady-state output $y = r$) and with the characteristic polynomial $s^2 + 2\zeta_0\omega_0 s + \omega_0^2$.

   (c) Compute the observability matrix for the system and show that the (linearized) system is observable.

   (d) Write down an observer for the system of the form $d\hat{x}/dt = \ldots$ and compute the estimator gain matrix so that the characteristic polynomial associated with the observer dynamics is again $s^2 + 2\zeta_e\omega_e s + \omega_e^2$.

   (e) Set $\omega_0 = 1$ and $\zeta_0 = 0.5$, and $\omega_e = 2$ and $\zeta_e = 0.5$. Compute the eigenvalues for the resulting closed-loop system when (i) state feedback is used ($u = -Kx$) and (ii) using feedback of the estimated state ($u = -K\hat{x}$). Note that the feedforward $k_r r$ can be ignored as it does not affect the eigenvalues.

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2. Åström and Murray, Exercise 6.10

   Assume that the A matrix is diagonalizable (the theorem is valid but hard to prove with a non-trivial Jordan form).

3. Åström and Murray, Exercise 6.12

   Download the file `bike_linmod.m` from the course web-site, which contains the parameters for the bicycle and generates the matrices M, C, K_0 and K_2 in Eq. (3.7) of the text.

   Find the controller gains corresponding to choosing the final pair of complex poles at $-1 \pm i$ as stated in the text, and also with these poles at $-2 \pm 2i$ and $-5 \pm 5i$. In addition to calculating the state feedback gains, solve for the reference gain k_r as well! When simulating the response to a step change in the desired steering angle (the "reference" value) of 0.002 rad, plot both the steering angle output δ and the torque command.

4. Consider the Whipple bicycle model given by equation (3.7) in Section 3.2. A state feedback for the system was designed in Exercise 6.12.

(a) Design an observer for the system with eigenvalues at -4, -20, and at $-2 \pm 2i$.

(b) Design an output feedback for the system using your observer from part (a) above and the first set of state feedback gains designed in the previous problem (with final eigenvalues at $-1 \pm 1i$), but with $u = -K\hat{x} + k_r r$ rather than $u = -Kx + k_r r$. Again, simulate the response to a step change in the reference value for the steering angle of 0.002 rad and plot both the steering angle output and torque command, and compare with the results obtained with full-state feedback. Plot the results with a perfect initial estimate $\hat{x}(0) = x(0)$, and also with a non-zero error in the estimated value of δ at time zero, e.g., $\widetilde{\delta}_0 = \hat{\delta}(0) - \delta(0) = 0.0002$.