CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

# CDS 101

D. MacMartin | **Problem Set #2** | Issued: 6 Oct 14
Fall 2014 | | Due: 15 Oct 14

**Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).**

1. Choose any *one* of the following systems, locate the equilibrium points for the system and indicate whether each is asymptotically stable, stable (but not asymptotically stable) or unstable. To determine stability, you can either use a phase portrait (if appropriate), analyze the linearization or simulate the system using multiple nearby initial conditions to determine how the state evolves.

   (a) *Nonlinear spring mass.* Consider a nonlinear spring mass system with dynamics

   $$m\ddot{q} = -k(q - aq^3) - c\dot{q},$$

   where $m = 1000$ kg is the mass, $k = 250$ kg/s$^2$ is the nominal spring constant, $a = 0.01$ represents the nonlinear "softening" coefficient of the spring and $c = 100$ kg/s is the damping coefficient. Note that this is very similar to the spring mass system we have studied in Section 2.2, except for the nonlinearity.

   (b) *Genetic toggle switch.* Consider the dynamics of two repressors connected together in a cycle. It can be shown (Exercise 2.9) that the normalized dynamics of the system can be written as

   $$\frac{dz_1}{d\tau} = \frac{\mu}{1 + z_2^n} - z_1 - v_1, \qquad \frac{dz_2}{d\tau} = \frac{\mu}{1 + z_1^n} - z_2 - v_2.$$

   where $z_1$ and $z_2$ represent scaled versions of the protein concentrations, $v_1$ and $v_2$ represent external inputs and the time scale has been changed. Let $\mu = 2.16$, $n = 2$ and $v_1 = v_2 = 0$.

   (c) *Congestion control of the Internet.* A simplified model for congestion control between $N$ computers connected by a router is given by the differential equation

   $$\frac{dx_i}{dt} = -b\frac{x_i^2}{2} + (b_{\max} - b), \qquad \frac{db}{dt} = \left(\sum_{i=1}^{N} x_i\right) - c,$$

   where $x_i \in \mathbb{R}$, $i = 1, \ldots, N$ are the transmission rates for the sources of data, $b \in \mathbb{R}$ is the current buffer size of the router, $b_{\max} > 0$ is the maximum buffer size and $c > 0$ is the capacity of the link connecting the router to the computers. The $\dot{x}_i$ equation represents the control law that the individual computers use to determine how fast to send data across the network and the $\dot{b}$ equation represents the rate at which the buffer on the router fills up. Consider the case where $N = 2$ (so that we have three states, $x_1$, $x_2$ and $b$) and take $b_{\max} = 1$ Mb and $c = 2$ Mb/s.

2. Åström and Murray, Exercise 4.3 (Pay attention to the range over which you plot the phase portrait, e.g., from 15 to 25 m/s captures the "interesting" part of the velocity state.)

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1. For *all* of the following systems, locate the equilibrium points for the system and indicate whether each is asymptotically stable, stable (but not asymptotically stable) or unstable. To determine stability, you can either use a phase portrait (if appropriate), analyze the linearization or simulate the system using multiple nearby initial conditions to determine how the state evolves.

   (a) *Nonlinear spring mass.* Consider a nonlinear spring mass system with dynamics

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2. Åström and Murray, Exercise 4.3 (Pay attention to the range over which you plot the phase portrait, e.g., from 15 to 25 m/s captures the "interesting" part of the velocity state.)

3. Åström and Murray, Exercise 4.4, changing second Lyapunov function to

$$V_2(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + \frac{b}{c-a}x_1)^2$$

   (change in sign in second term if you are using the textbook; no change if you are using the online pdf).

4. There are two containers, one contains 1 liter of wine, and one contains 1 liter of water. An amount $\alpha < 1$ of liquid is transferred from one container to the other, the results well-mixed, and the same amount is then transferred back and mixed. Define this pair of transfers as a single "step", so that after each step, both containers still hold one liter of liquid. Is there ever the same amount of water as there is wine in each container (after $N$ steps)? Does the ratio of wine to water in each container approach a unique fixed value? If so, how does the choice of $\alpha$ determine how rapidly the final ratio is reached?