CDS 101/110: Lecture 9-1
PID Control

Douglas G. MacMartin

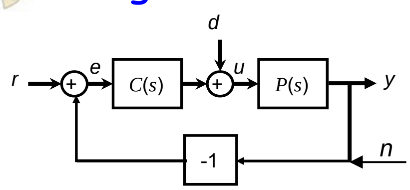
Goals:

- Show how to use PID (Proportional + Integral + Derivative) feedback to achieve a performance specification

Reading:

- Åström and Murray, *Feedback Systems,* Ch 10
- *Advanced*: Lewis, Chapters 12-13

Design based on loop transfer function



$$L(s) = P(s)C(s)$$

$$H_{er} = \frac{1}{1+L} \qquad H_{yr} = \frac{L}{1+L}$$

$$H_{yd} = \frac{P}{1+L} \qquad H_{yn} = \frac{-L}{1+L}$$

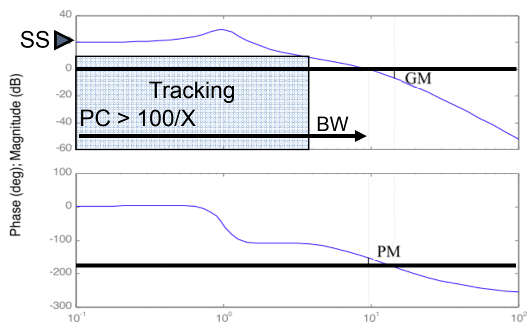
- Stability depends only on $L = PC$
  - Robustness requires reasonable gain and phase margin
- Performance depends (mostly) on $L = PC$
  - When L is large, tracking performance and disturbance rejection is good
  - When L is small, sensor noise rejection is good, actuator response is small.
  - Typically care about the tracking and disturbance response at low frequencies
  - If gain or phase margin is small, tend to get large overshoot and ringing

Frequency Domain Performance Specifications

- Specify bounds on the loop transfer function to guarantee desired performance
  - Steady state error: $H_{er}(0) = \dfrac{1}{(1 + L(0))} \simeq 1/L(0)$

    ⇒ sets zero frequency ("DC") gain
  - Tracking: Error less than X% up to frequency ω

    ⇒ Determines gain bound |1+L|>100/X



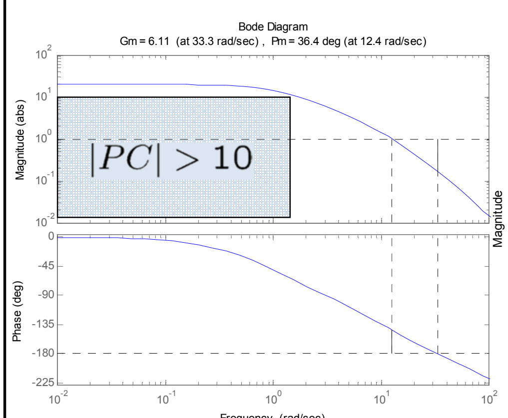
- Bandwidth:
  - assuming ~90˚ phase margin

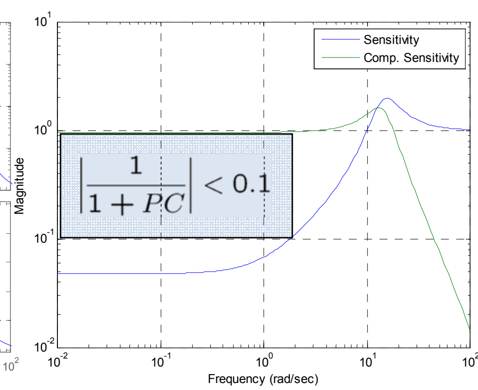
$$\frac{L}{1+L}(i\omega_c) = \left|\frac{1}{1+i}\right| = \frac{1}{\sqrt{2}}$$

  - In general, loop crossover frequency will be close to the bandwidth

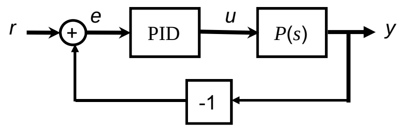
Example:

- Suppose $L(s) = \dfrac{20}{(s+1)(s/10+1)(s/100+1)}$





Overview: PID control

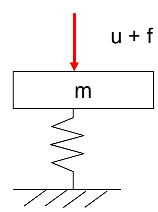


$$u = k_p e + k_i \int e \, \mathrm{d}t + k_d \dot{e}$$

$$C(s) = \frac{k_i}{s} + k_p + k_d s$$

- Intuition
  - Proportional term: provides inputs that correct for “current” errors
  - Integral term: ensures steady state error goes to zero
  - Derivative term: provides “anticipation” of upcoming changes
- A bit of history on “three term control”
  - First appeared in 1922 paper by Minorsky: “Directional stability of automatically steered bodies” under the name “three term control”
  - Also realized that “small deviations” (linearization) could be used to understand the (nonlinear) system dynamics under control
- Utility of PID
  - PID control is most common feedback structure in engineering systems
  - For many systems, only need PI or PD (special case)
  - Many tools for tuning PID loops and designing gains

Time-domain motivation



- Mass-spring system: $m\ddot{z} + c\dot{z} + kz = u + f$
- PD control: $u = -(k_p z + k_d \dot{z})$
- Closed-loop: $m\ddot{z} + (c + k_d)\dot{z} + (k + k_p)z = f$
  - Derivative gain acts like increasing damping
    - Increases system stability (greater phase margin)
  - Proportional gain acts like increasing stiffness
    - No matter how large the stiffness, still a non-zero response to disturbance force
  - Steady-state (for constant disturbance force f)

$$\dot{z} = 0 \quad \Rightarrow \quad \lim_{t\to\infty} z(t) = \frac{1}{k + k_p} f, \text{ and } \lim_{t\to\infty} u(t) = -\frac{k_p}{k + k_p} f$$

- Integral control: $\dot{q} = z$

$$u = -(k_i q + k_p z + k_d \dot{z})$$

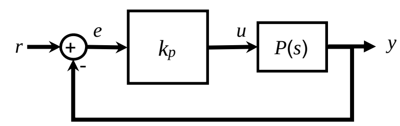
  - Steady-state (assuming stability) then for constant f

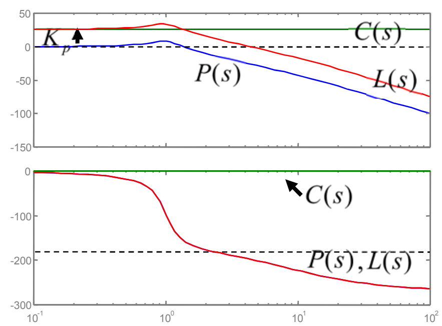
$$\dot{q} = 0 \quad \Rightarrow \quad \lim_{t\to\infty} z = 0, \text{ and } \lim_{t\to\infty} u(t) = -f$$

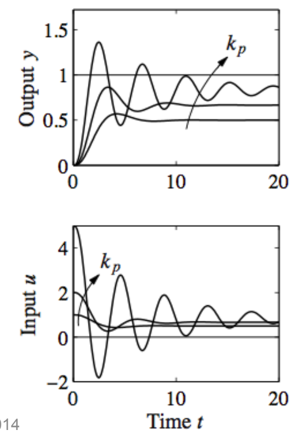
Proportional Feedback

$k_p > 0$ if $P(0) > 0$

- Simplest controller choice: $u = k_p e$
  - Effect: lifts gain with no change in phase
  - Good for plants with low phase up to desired bandwidth
  - Bode: shift gain up by factor of k_p
  - Step response: better steady state error, but with decreasing stability

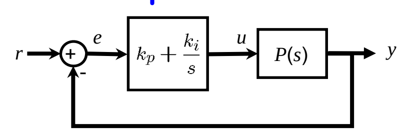




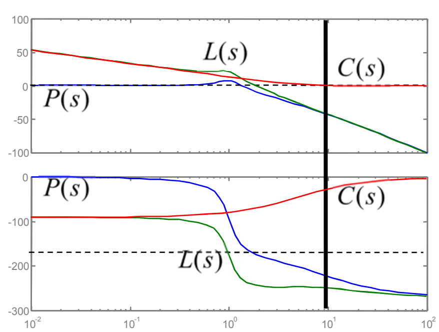


Proportional + Integral Compensation

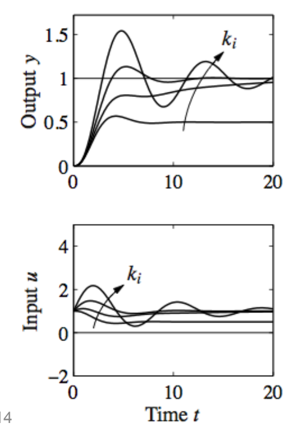
- Use to eliminate steady state error
  - Effect: lifts gain at low frequency
  - Gives *zero* steady state error
  - Bode: infinite SS gain + phase lag
  - Step response: zero steady state error, with smaller settling time, but more overshoot



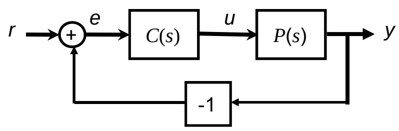
$k_p > 0$, $k_i > 0$



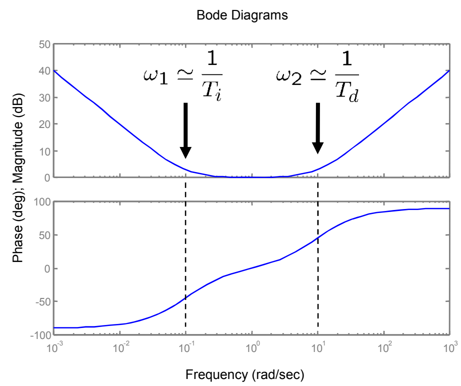
$\omega_z = k_i / k_p$

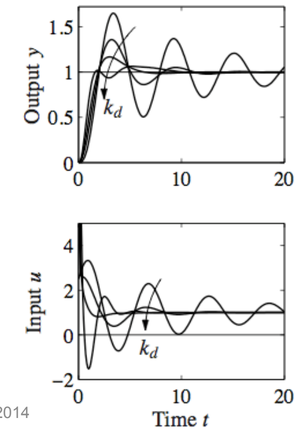


Proportional + Integral + Derivative (PID)

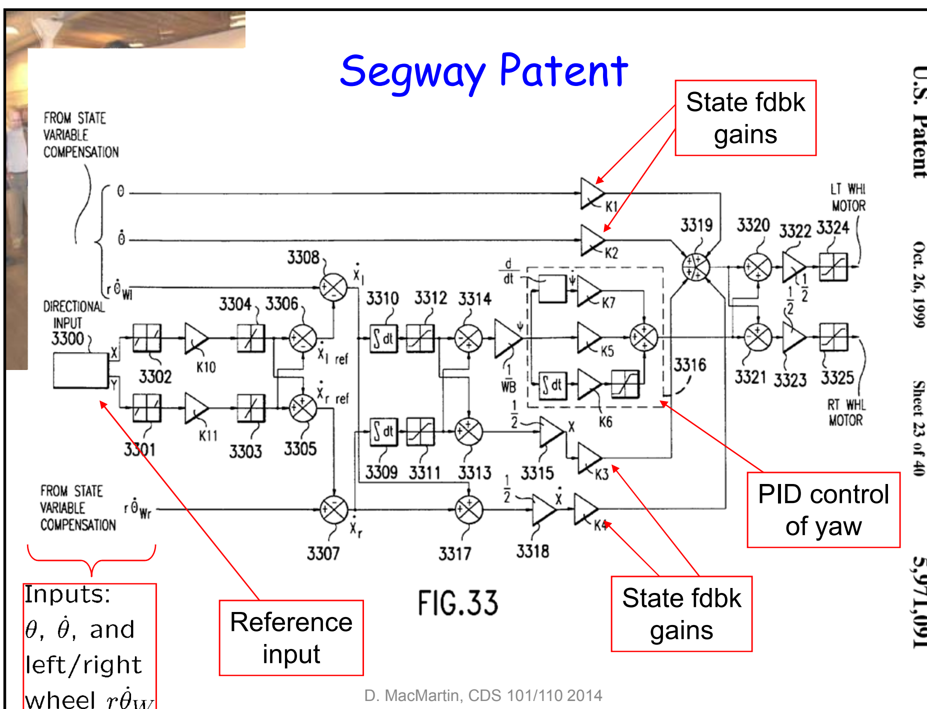


$$C(s) = k_p + k_i \frac{1}{s} + k_d s$$
$$= k\left(1 + \frac{1}{T_i s} + T_d s\right)$$
$$= (kT_d)\,\frac{(s+\alpha_i)(s+\alpha_d)}{s}$$

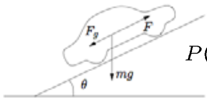




Segway Patent

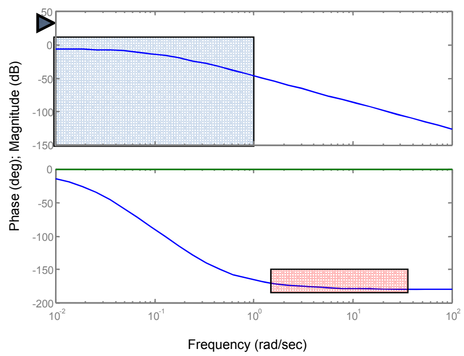


Example: Cruise Control using PID - Specification



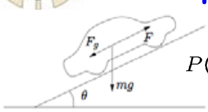
$$P(s) = \frac{1/m}{s + b/m} \times \frac{r}{s+a}$$

$b = 50,\ m = 1000,\ a = 0.2,\ r = 5$



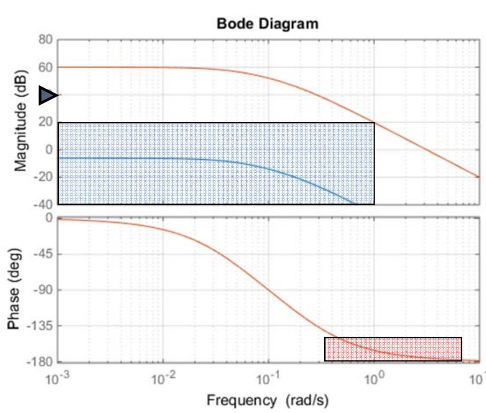
- Performance Specification
  - $\leq$ 1% steady state error (*ALT: zero SS err*)
    - $\Rightarrow$ Zero frequency gain > 100
  - $\leq$ 10% tracking error up to 1 rad/sec
    - $\Rightarrow$ Gain > 10 from 0-1 rad/sec
  - $\geq$ 45° phase margin
    - $\Rightarrow$ Gives good relative stability
    - $\Rightarrow$ Provides robustness to uncertainty
    - $\Rightarrow$ But overshoot will be ~25%
- Observations
  - Purely proportional gain won't work: to get gain above desired level will not leave adequate phase margin
  - Need to increase the phase from ~0.5 to 2 rad/sec and increase gain as well

Example: Cruise Control using PID: Kp



$$P(s) = \frac{1/m}{s + b/m} \times \frac{r}{s+a}$$

$b = 50,\ m = 1000,\ a = 0.2,\ r = 5$



- With only k_p: satisfy
  - $\leq$ 10% tracking error up to 1 rad/sec
    - $\Rightarrow$ Gain > 10 from 0-1 rad/sec
- Plant magnitude at 1 rad/sec:

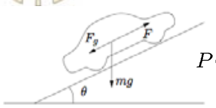
$$|P(1i)| = \frac{r}{\sqrt{m^2+b^2}\sqrt{1+a^2}}$$

  - Choose

$$k_p = 10\frac{\sqrt{m^2+b^2}\sqrt{1+a^2}}{r} \simeq 10\frac{m}{r} \simeq 2000$$

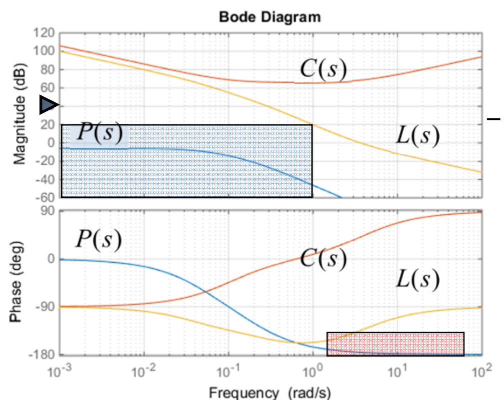
  - (Actually, need k_p>2042)
  - Steady-state tracking error determined by $k_p P(0)$ = 1000; requirement is 100
  - Note inadequate phase margin

Example: Cruise Control using PID - Design



$$P(s) = \frac{1/m}{s + b/m} \times \frac{r}{s+a}$$

$b = 50,\ m = 1000,\ a = 0.2,\ r = 5$

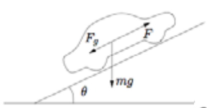


- Use proportional gain to give desired tracking performance
- (*ALT*) If specification required zero steady state error, use integral gain
  - Place the zero corresponding to (k_i+$k_p s$) at ~0.1 → minimal net phase effect of zero and integrator pole at crossover frequency
  - E.g., choose k_i=0.1k_p
- Use derivative action to increase phase lead in the cross over region
  - Loop crossover with only kp is roughly 3 rad/sec,
  - Place the zero corresponding to (k_p+$k_d s$) at ~4 → roughly 45° phase margin
  - E.g., choose k_d=k_p/4

$$C(s) = 2000\frac{s^2/4 + 1s + 0.1}{s} = 2000 + \frac{200}{s} + 500s$$

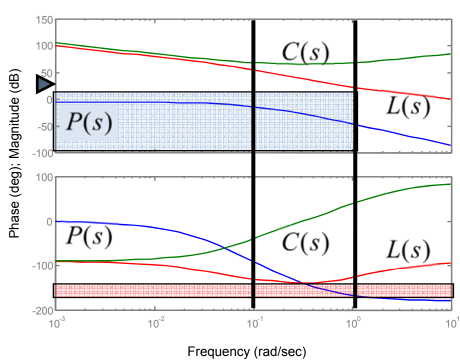
Verify with Nyquist + Gang of 4

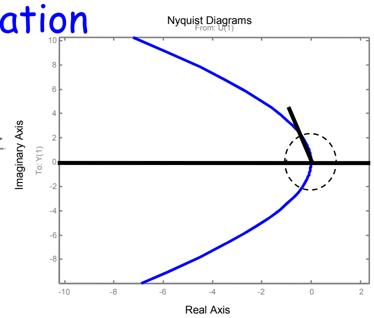
Example: Cruise Control using PID - Verification

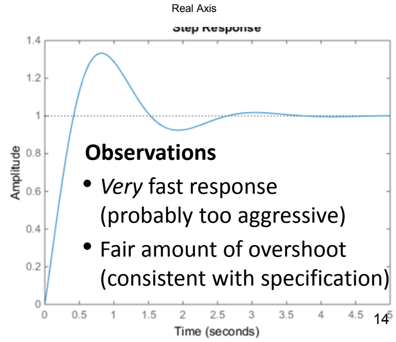


$$P(s) = \frac{1/m}{s + b/m} \times \frac{r}{s+a}$$

$$C(s) = 2000\frac{s^2/4 + 1s + 0.1}{s}$$



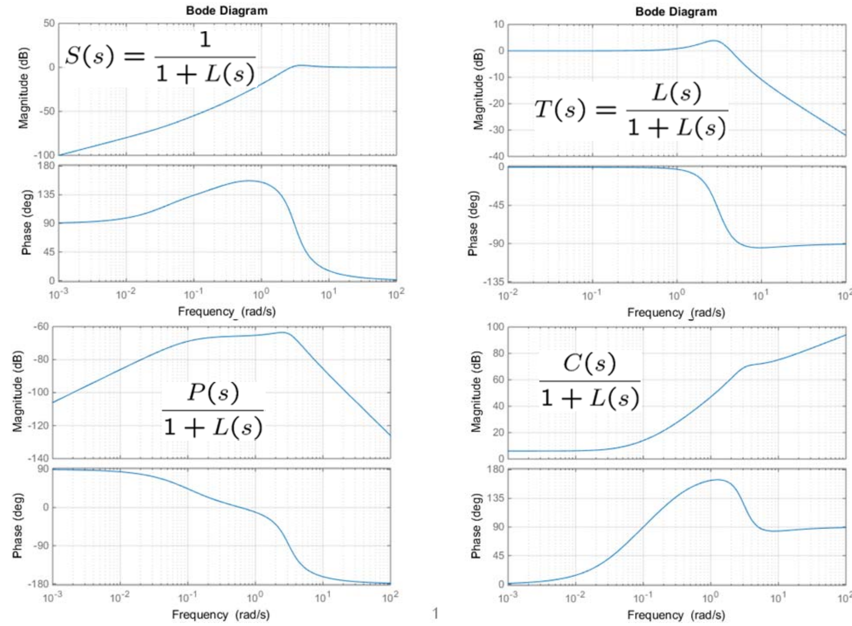




Observations

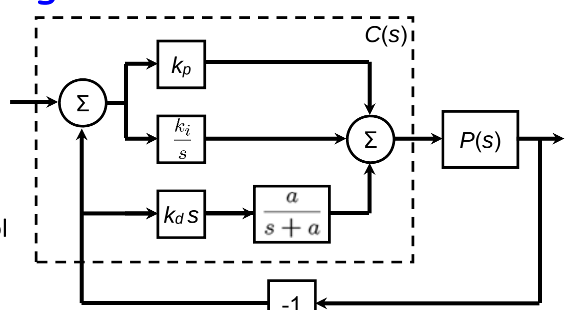
- *Very* fast response (probably too aggressive)
- Fair amount of overshoot (consistent with specification)

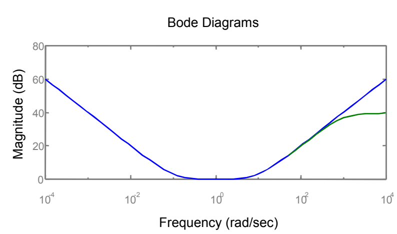
Gang of Four… is this ok?



Implementing Derivative Action

- Problems with derivatives
  - High frequency noise amplified by derivative term
  - Step inputs in reference can cause large inputs
  - Shows up in Gang of Four...
- Solution: modified PID control
  - Use high frequency rolloff in derivative term
    - first order filter will give finite gain at high frequency
    - use higher order filter if needed
  - Don't feed reference signal through derivative block
    - Useful when reference has unwanted high frequency content
    - Alternative solution: reference shaping via two DOF design (*F*(*s*) block)
  - Many other variations (see AM08 + refs)





Comparison with Lead/Lag

- With derivative roll-off (so that controller is proper), and
- Integrator "leakage" (so that pole is not strictly at origin), then

$$C(s) = \frac{k_i}{s+\epsilon} + k_p + \frac{k_d s}{1+s/a}$$

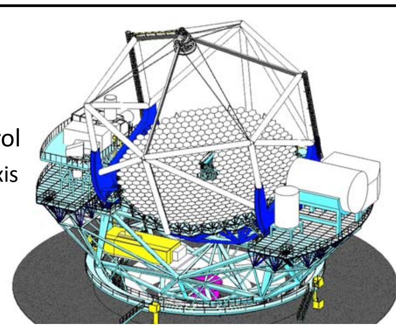
$$= k\frac{(s+b)(1+s/c)}{(s+\epsilon)(1+s/a)} \qquad b \simeq \frac{k_i}{k_p} \quad c \simeq \frac{k_p}{k_d} \qquad \epsilon < b < c < a$$

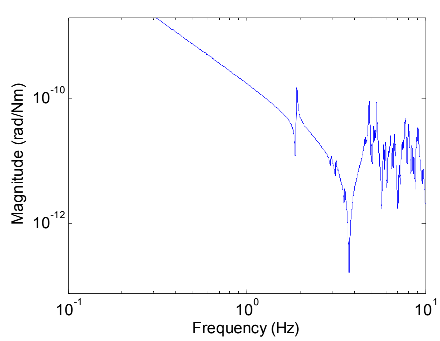
$$= k\underbrace{\left(\frac{s+b}{s+\epsilon}\right)}_{\text{Lag controller}}\underbrace{\left(\frac{1+s/c}{1+s/a}\right)}_{\text{Lead controller}}$$

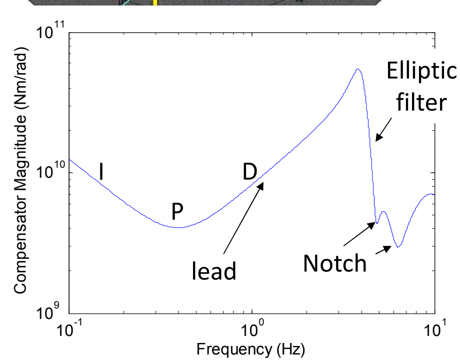
- Different knobs, similar result
- Note, not all systems and control objectives can be controlled by this controller structure… may not need all these terms, or may need additional tools

Example

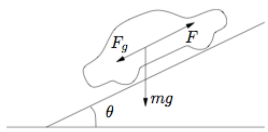
- Thirty meter telescope pointing control
  - Input is drive torque about elevation axis
  - Output is (collocated) rotation from encoder output
  - Need notch filters to compensate for resonant peaks

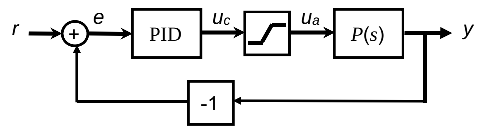




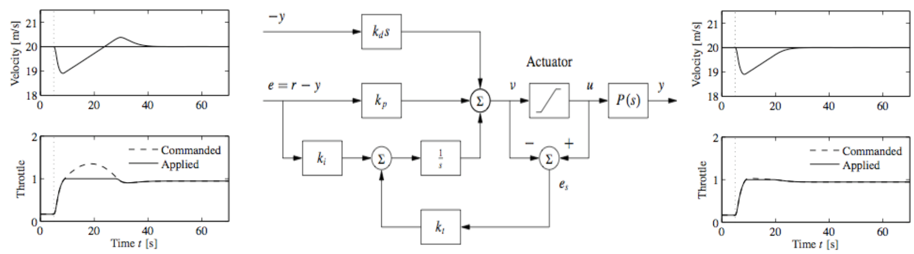


Windup and Anti-Windup Compensation





- Problem
  - Limited magnitude input (saturation)
  - Integrator "winds up" ⇒ overshoot
- Solution
  - Compare commanded input to actual
  - Subtract off difference from integrator

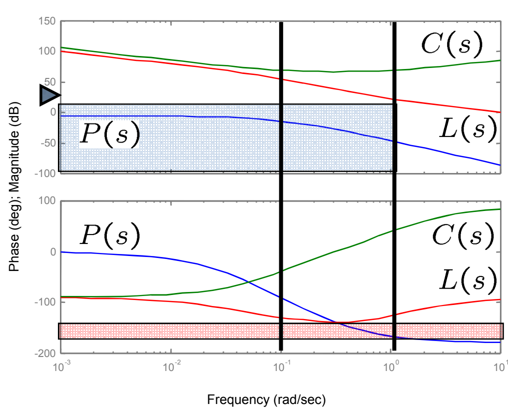


(a) Windup (b) Anti-windup

Summary: Frequency Domain Design using PID

- Loop Shaping for Stability & Performance
  - Steady state error, bandwidth, tracking

$$H_{ue}(s) = k_p + k_i \frac{1}{s} + k_d s$$



- Main ideas
  - Performance specs give bounds on loop transfer function
  - Use controller to shape response
  - Gain/phase relationships constrain design approach
  - Standard compensators: proportional, PI, PID