CDS 101/110: Lecture 8-1
Frequency Domain Design

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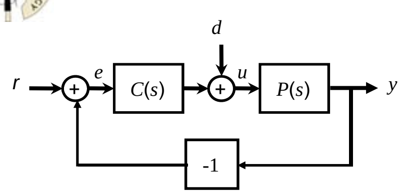
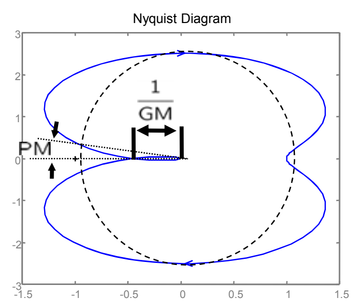
Goals:
- Describe canonical control design problem and standard performance measures
- Show how to use "loop shaping" to achieve a performance specification
- Work through a detailed example of a control design problem

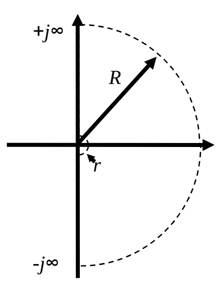
Reading:
- Åström and Murray, *Feedback Systems,* Ch 11
- *Advanced*: Lewis, Chapter 12

Last Week: Loop Analysis





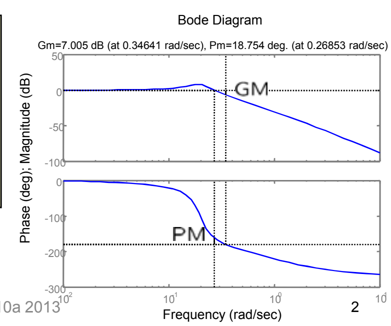
- Nyquist criteria for loop stability
- Gain, phase margin for robustness



Thm (Nyquist).

P # RHP poles of *L*(*s*)
N # CW encirclements
Z # RHP zeros of 1+*L*

$Z = N + P$



Sensitivity

- From Rowley & Battin, *Fundamentals & Applications of Modern Flow Control*, Ch 5
- Example plotted is:

$$\frac{20}{(s+1)(s+2)(s+3)}$$

- Distance from -1 impacts performance as well as robustness

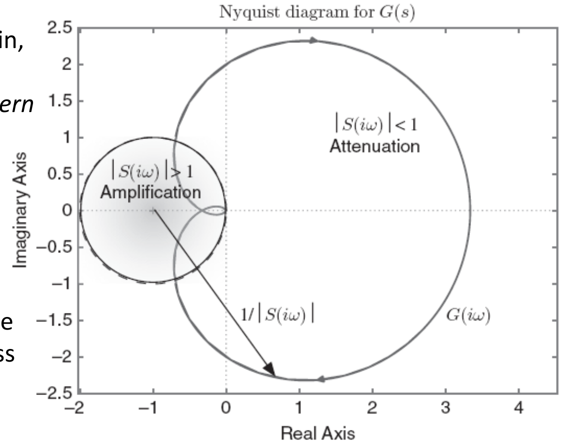
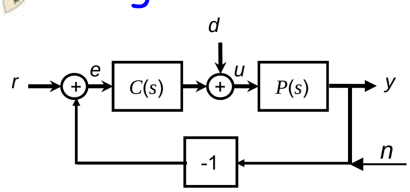


Fig. 4 Nyquist plot of the loop gain $G(s) = P(s)C(s)$ for the system (29). For frequencies for which $G(i\omega)$ enters the unit circle centered about the -1 point, disturbances are amplified and, for frequencies for which $G(s)$ lies outside this circle, disturbances are attenuated relative to open-loop.

Design based on loop transfer function



$$L(s) = P(s)C(s)$$

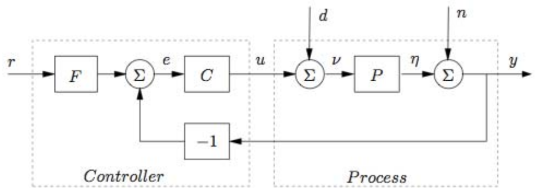
$$H_{er} = \frac{1}{1+L} \qquad H_{yr} = \frac{L}{1+L}$$

$$H_{yd} = \frac{P}{1+L} \qquad H_{yn} = \frac{-L}{1+L}$$

- Stability depends only on *L = PC*
  - Robustness requires reasonable gain and phase margin
- Performance depends *(mostly)* on *L = PC*
  - When L is large, tracking performance and disturbance rejection is good
  - When L is small, sensor noise rejection is good, actuator response is small.
  - Typically care about the tracking and disturbance response at low frequencies
  - If gain or phase margin is small, tend to get large overshoot and ringing
- Definitions:

$$S(s) = \frac{1}{1+L} := \text{Sensitivity} \qquad T(s) = \frac{L}{1+L} = \text{complementary sensitivity}$$

Two Degree of Freedom Design and "Gang of four" transfer functions



Typical design procedure

- Design *C* to balance all requirements
- Design *F* to improve response to reference

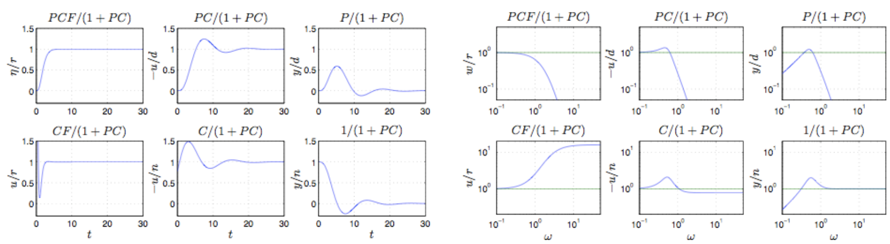
- For F=1:

$$S = \frac{1}{1+PC}$$ Sensitivity function

$$T = \frac{PC}{1+PC}$$ Complementary sensitivity

$$PS = \frac{P}{1+PC}$$ Load sensitivity

$$CS = \frac{C}{1+PC}$$ Noise sensitivity

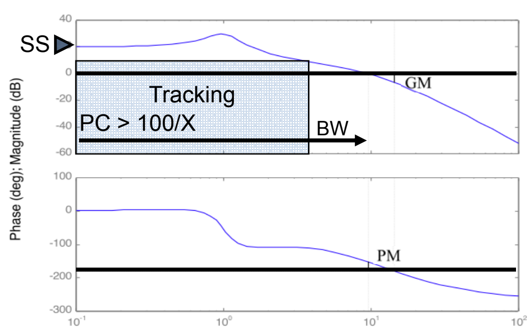


Frequency Domain Performance Specifications

- Specify bounds on the loop transfer function to guarantee desired performance
  - Steady state error: $H_{er}(0) = \dfrac{1}{(1+L(0))} \simeq 1/L(0)$

    ⇒ sets zero frequency ("DC") gain ▶
  - Tracking: Error less than X% up to frequency ω

    ⇒ Determines gain bound |1+L|>100/X



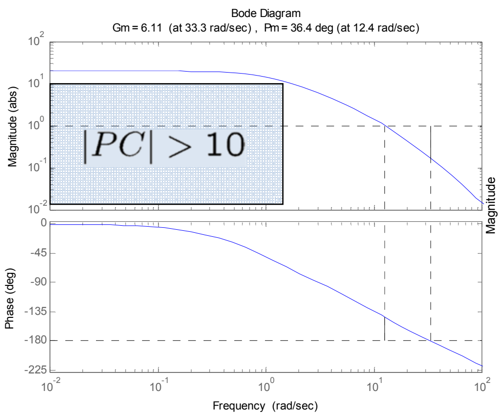
- Bandwidth:
  - assuming ~90˚ phase margin

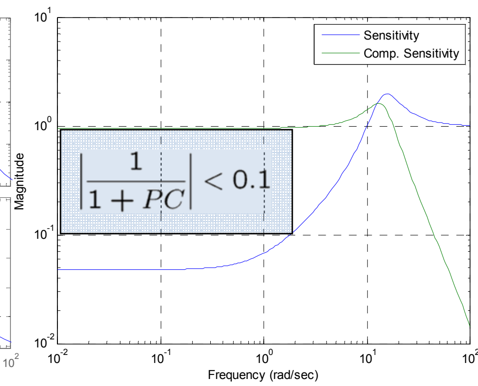
$$\frac{L}{1+L}(i\omega_c) = \left|\frac{1}{1+i}\right| = \frac{1}{\sqrt{2}}$$

⇒ sets loop crossover frequency

Example:

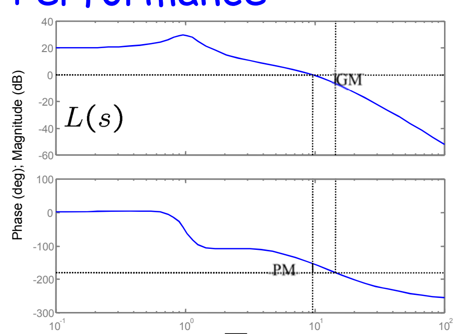
- Suppose $L(s) = \dfrac{20}{(s+1)(s/10+1)(s/100+1)}$

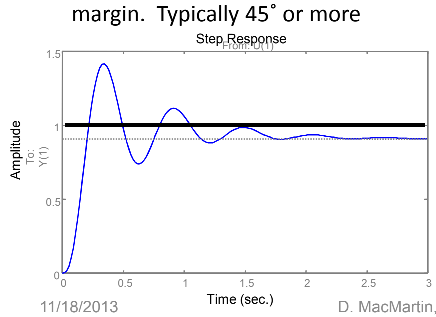


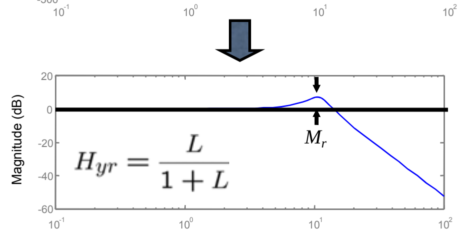


Relative Stability: Robustness and Performance

- Loop transfer function close to -1 gives poor robustness *and* poor performance:
  - System can be stable but still have bad response at certain frequencies
  - Typically occurs if system has low phase margin ⇒ get resonant peak in closed loop ⇒ overshoot; poor step response
  - Solution: specify minimum phase margin. Typically 45˚ or more

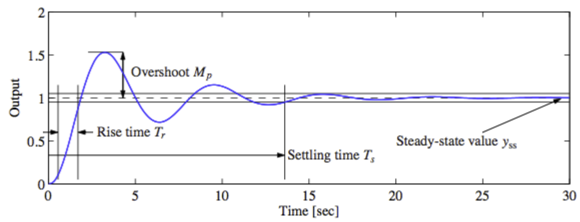


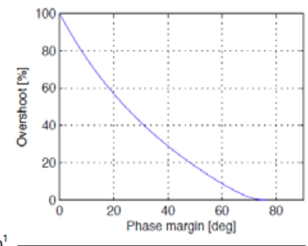




Time-domain specifications → frequency-domain

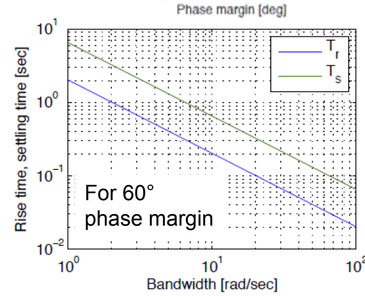
- Time-domain specifications (rise-time, overshoot, settling time…) can be related to frequency-domain for a second order system





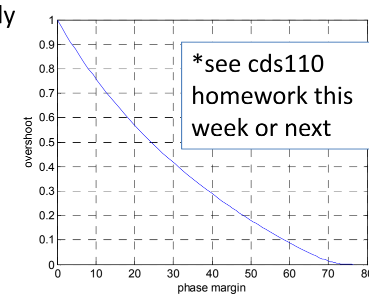
- Second-order system (see homework):

$$L(s) = \frac{k}{s^2 + bs} \qquad H_{yr} = \frac{k}{s^2 + bs + k}$$



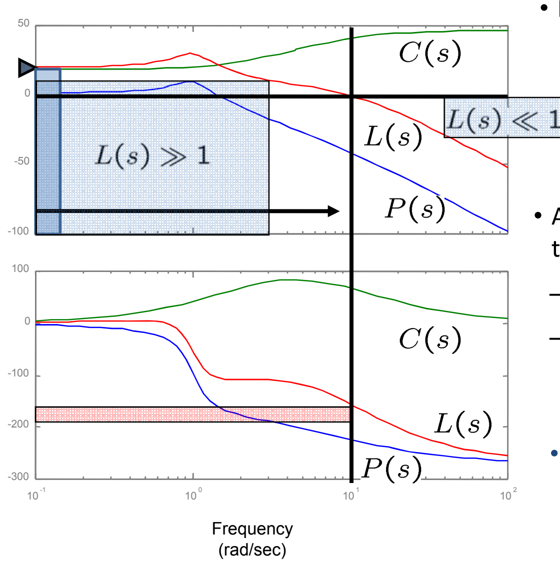
Summary of specifications

- Steady-state tracking error < X — |L(0)| > 1/X
- Tracking error < Y up to frequency f_t Hz — $|L(i\omega)| > 1/Y$ for $\omega < 2\pi f_t$
- Bandwidth of ω_b rad/sec — Roughly, $|L(i\omega_b)|=1$
  - Strictly, need to specify whether this means |T|= −3 dB or |S|= −3 dB
- Overshoot < Z — Phase margin > f(Z)*
  - Can evaluate after design and confirm
- May specify phase and gain margins directly
  - Typically gain margin of 2 (i.e. 6 dB)
  - Typically phase margin 30-60 degrees



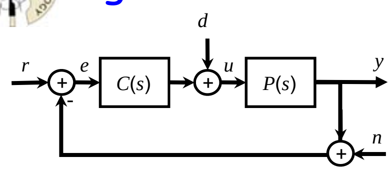
*see cds110 homework this week or next

Graphical Overview of Loop Shaping



- Performance specification
  - Steady state error
  - Tracking error
  - Bandwidth
  - Relative stability
- Approach: "shape" loop transfer function using $C(s)$
  - $P(s)$ + specifications given
  - $L(s) = P(s)\, C(s)$
    - Use $C(s)$ to choose desired shape for $L(s)$
- **Important: can't set gain and phase independently**
  - **Shallow slope at cross-over for sufficient phase margin**

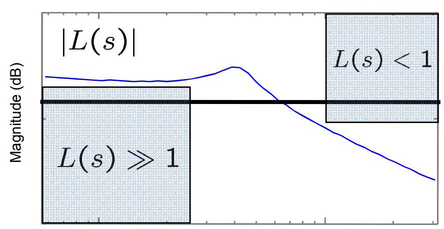
Algebraic Constraints on Performance



$$H_{er} = \frac{1}{1+PC} =: S$$ Sensitivity function

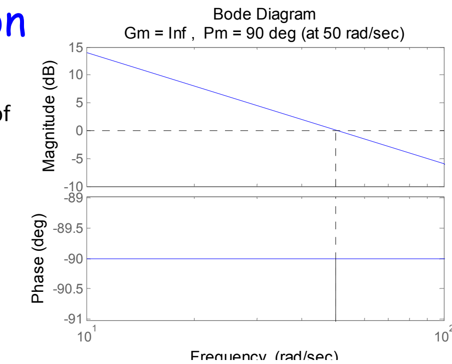
$$H_{yn} = \frac{PC}{1+PC} =: T$$ Complementary sensitivity function

- Goal: keep S & T small
  - S small ⇒ low tracking error
  - T small ⇒ good noise rejection (and robustness [CDS 110b])
- Problem: S + T = 1
  - Can't make *both* S & T small at the same frequency
  - Solution: keep S small at low frequency and T small at high frequency
  - Loop gain interpretation: keep L large at low frequency, small at high freq.



- Transition between large gain and small gain complicated by stability (phase margin)

Process Inversion



- Simple trick: invert out process
  - Write all performance specs in terms of the desired loop transfer function
  - Choose $L(s)$ to satisfies specifications
  - Choose controller by inverting $P(s)$

$$C(s) = L(s)/P(s)$$

- Pros
  - Very easy design process
  - $L(s)$ = k/s often works very well
  - Can be used as a first cut, with additional shaping to tune design
- Cons
  - High order controllers (at least same order as the process you are controlling)
  - Requires "perfect" model of your process (since you are inverting it)
  - *Does not work if you have right half plane poles or zeros* (not internally stable)

$$S = \frac{1}{1+PC} \qquad T = \frac{PC}{1+PC} \qquad PS = \frac{P}{1+PC} \qquad CS = \frac{C}{1+PC}$$

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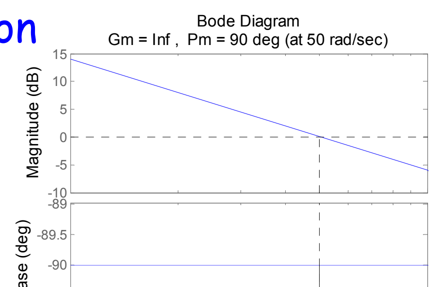

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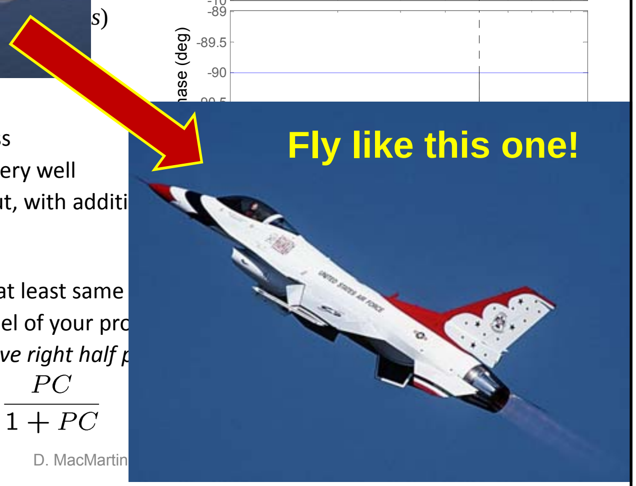
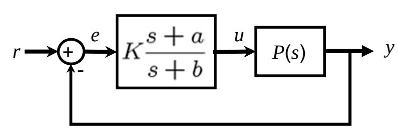


(Better) Loop Shaping Design Tools

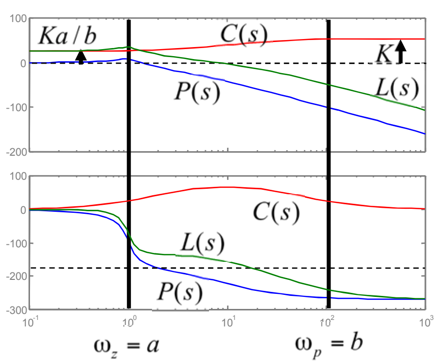
- Start with plant, and performance goals
  - If plant is not stable, find starting controller to stabilize; more on Wednesday…
- Increase gain to meet performance…
  - Include an integrator if you want zero steady-state error
- Is it stable? Do you have enough phase margin?
- Tools:
  - Reduce gain until adequate margin
  - Proportional + Integral + Differential (PID): $\frac{k_i}{s} + k_p + k_d s$
    - Same basic idea (find appropriate gain and phase to satisfy performance and robustness), different knobs.
  - **Lead compensator to add phase near crossover:** $K\frac{s+a}{s+b} \quad a < b$
  - Lag compensator to increase low frequency gain: $K\frac{s+a}{s+b} \quad a > b$
- ***This is a design; there is no "right" answer!***

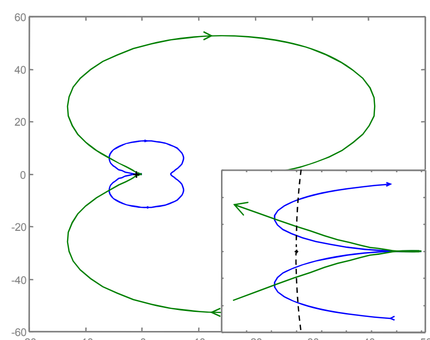
Lead compensation

- Use to increase phase in frequency band
  - Effect: lifts phase by increasing gain at high frequency
  - *Very* useful controller; increases PM
  - Bode: add phase between zero and pole
  - Nyquist: increase phase margin



$a < b \qquad K > 0$





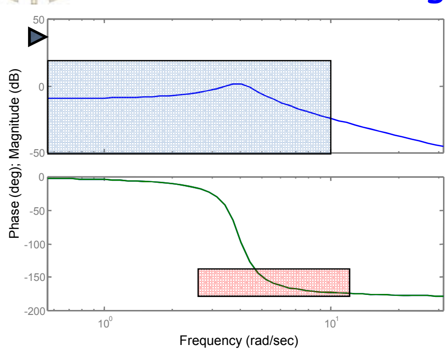
Example: Control of Vectored Thrust Aircraft

- System description
  - Vector thrust engine attached to wing
  - Inputs: fan thrust, thrust angle (vectored)
  - Outputs: position and orientation
  - States: *x*, *y*, θ + derivatives
  - Dynamics: flight aerodynamics

Control approach

- Design "inner loop" control law to regulate pitch (θ) using thrust vectoring
- Second "outer loop" controller regulates the position and altitude by commanding the pitch and thrust
- Basically the same approach as aircraft control laws

Performance Specification and Design Approach



$$P(s) = \frac{r}{Js^2 + ds + mg\ell}$$

Performance Specification

- $\leq$ 1% steady state error
  - Zero frequency gain > 100
- $\leq$ 10% tracking error up to 10 rad/sec
  - Gain > 10 from 0-10 rad/sec
- $\geq$ 45˚ phase margin
  - Gives good relative stability
  - Provides robustness to uncertainty

- Design approach
  - If choose C(s)=K, then poor phase margin
  - Add phase lead in 5-50 rad/sec range
  - Increase the gain to achieve steady state and tracking performance specs

$$C(s) = K\frac{s+a}{s+b}$$

$a = 25$

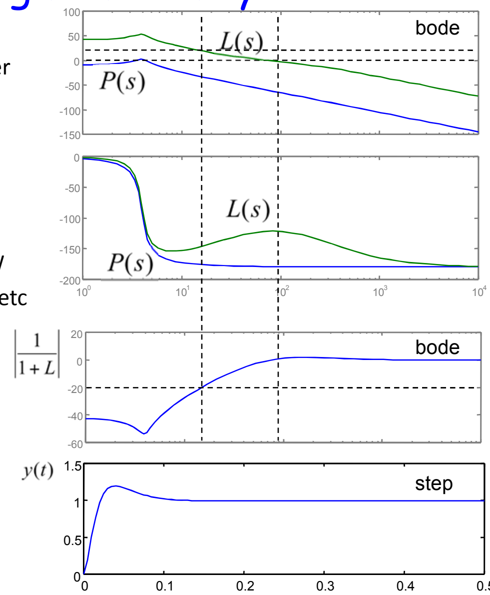
$b = 300$

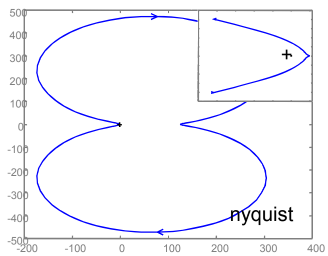
$K = 15 \times 300$

Control Design and Analysis

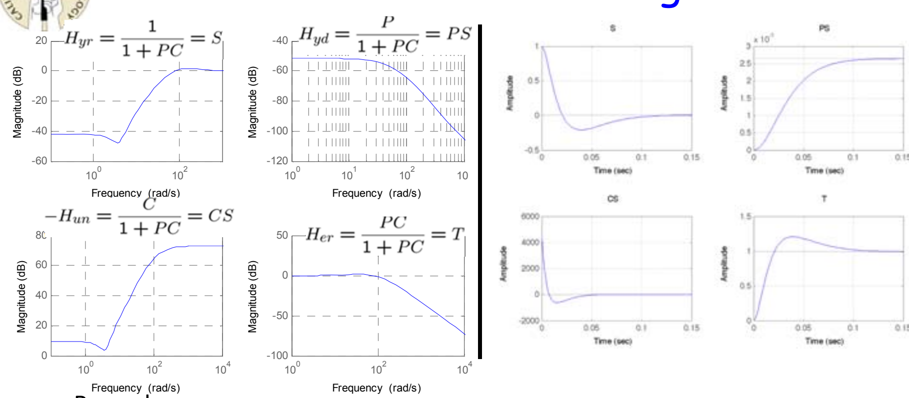
- Select parameters to satisfy specs
  - Place phase lead in desired crossover region (given by desired BW)
  - Phase lead peaks at $\omega = \sqrt{ab}$
  - Maximum phase depends on pole/zero ratio:

    $$\phi_{\max} = 90^\circ - 2\tan^{-1}\sqrt{a/b}$$
  - Set gain as needed for tracking + BW
  - Verify controller using Nyquist plot, etc





Control Verification: Gang of 4



- Remarks
  - Check each transfer function to look for peaks, large magnitude, etc
  - Example: Noise sensitivity function (*CS*) has very high gain; step response verifies poor step response
  - Implication: controller amplifies noise at high frequency ⇒ will generate *lots* of motion of control actuators (flaps)
  - Fix: roll off the loop transfer function faster (high frequency pole)

Lead & Lag Controller

- Lead:

$$K(s) = \frac{s+a}{s+b}, \qquad a < b$$

- Adds phase, maximum phase ϕ_m added at

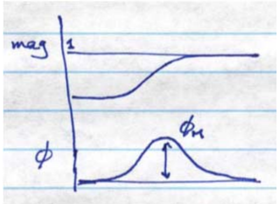
$$\omega = \sqrt{ab} \qquad \phi_m = 90° - 2\tan^{-1}\sqrt{a/b}$$

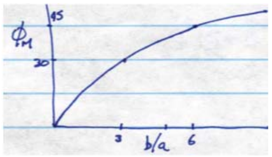
E.g.

ϕ_m	b/a
30°	~3
45°	~6
60°	~14

or

$$K(s) = \frac{\alpha T s + 1}{T s + 1}$$





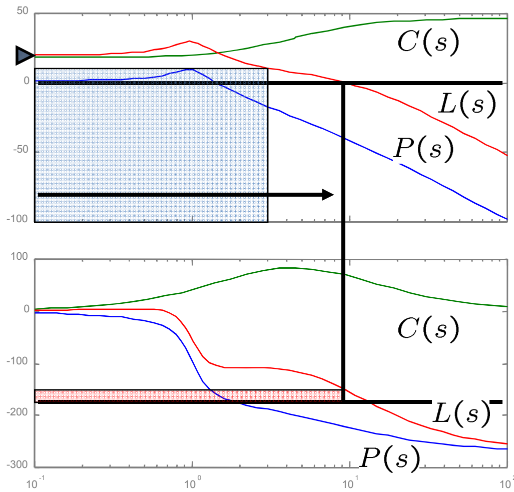
- Lag:

$$K(s) = \frac{s+a}{s+b}, \qquad b < a < \omega_c$$

a/b = increase in error constant
Use for steady-state performance

Summary: Loop Shaping

- Loop Shaping for Stability & Performance
  - Steady state error, bandwidth, tracking



Main ideas

- Performance specs give bounds on loop transfer function
- Use controller to shape response
- Gain/phase relationships constrain design approach
- Lead compensator useful to add phase