CDS 101/110a: Lecture 5-1
State Estimation

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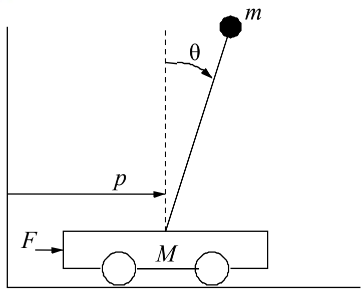
- Goals:
  - Introduce state estimator (aka observer)
  - Provide examples of state estimation in the context of closed loop design
  - Begin frequency-domain analysis
- Reading:
  - Åström and Murray, Feedback Systems, Sections 7.1-7.3

Observability

- Definition: A dynamical system of the form: $\dot{x} = f(x,u)$, $y = h(x,u)$
- is observable if for any T > 0 it is possible to determine the state of the system x(T) through measurements of y(t) and u(t) on the interval [0,T]
- Remarks
  - Observability must respect causality: only get to look at past measurements
  - Each initial condition must generate a unique output *y*
  - Start with ignoring noise, disturbances $\Rightarrow$ estimate exact state
- Test to check observability for linear system: $\dot{x} = Ax + Bu$, $y = Cx$
- Theorem: A linear system is observable iff the observability matrix W_o is full rank, where:

$$W_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Example #1: Linearized pendulum on a cart

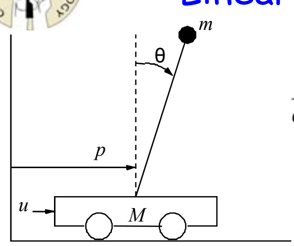


- Question: can we determine the state of the system $(p, \dot{p}, \theta, \dot{\theta})$ from measuring position p? How about from θ?
- Approach: look at the linearization around the upright position (good approximation to the full dynamics if θ remains small)

$$\frac{d}{dt}\begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2\ell^2 g}{M_t J_t - m^2\ell^2} & \frac{-cJ_t}{M_t J_t - m^2\ell^2} & \frac{-\gamma J_t \ell m}{M_t J_t - m^2\ell^2} \\ 0 & \frac{M_t m g \ell}{M_t J_t - m^2\ell^2} & \frac{-c\ell m}{M_t J_t - m^2\ell^2} & \frac{-\gamma M_t}{M_t J_t - m^2\ell^2} \end{bmatrix} \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{J_t}{M_t J_t - m^2\ell^2} \\ \frac{\ell m}{M_t J_t - m^2\ell^2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$$

Example #1, con't: Linearized pendulum on a cart



$$\frac{d}{dt}\begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2 l^2 g}{\mu} & \cancelto{0}{\frac{-cJ_t}{\mu}} & \cancelto{0}{\frac{-\gamma J_t l m}{\mu}} \\ 0 & \frac{M_t m g l}{\mu} & \cancelto{0}{\frac{-clm}{\mu}} & \cancelto{0}{\frac{-\gamma M_t}{\mu}} \end{bmatrix} \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{J_t}{\mu} \\ \frac{lm}{\mu} \end{bmatrix} u$$

$$\mu = M_t J_t - m^2 l^2$$

- Simplify by setting $c, \gamma = 0$

- Observability matrix for C = [1 0 0 0]:

$$W_o = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{m^2\ell^2 g}{\mu} & 0 & 0 \\ 0 & 0 & 0 & \frac{m^2\ell^2 g}{\mu} \end{bmatrix} \end{array} \begin{array}{l} C \\ CA \\ CA^2 \\ CA^3 \end{array}$$

- Full rank $\Rightarrow$ observable
- How about with C = [0 1 0 0] ?

$$W_o = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{M_t m g \ell}{\mu} & 0 & 0 \\ 0 & 0 & 0 & \frac{M_t m g \ell}{\mu} \end{bmatrix}$$

State Estimation: Full Order Observer

- Given that a system is observable, how do we actually estimate the state?
  - Key insight: if current estimate is correct, follow the dynamics of the system

$$\dot{x} = Ax + Bu \qquad \dot{\hat{x}} = \underbrace{A\hat{x} + Bu}_{\text{prediction (copy of dynamics)}} + \underbrace{L(y - C\hat{x})}_{\text{correction (based on output error)}}$$
$$y = Cx$$

  - Modify the dynamics to correct for error based on a linear feedback term
  - L is the observer gain matrix; determines how to adjust the state due to error
  - Look at the error dynamics for $\tilde{x} = x - \hat{x}$ to determine how to choose L:

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = Ax + Bu - (A\hat{x} + Bu + LC(x - \hat{x})) = (A - LC)\tilde{x}$$

- **Thm:** If the pair (A, C) is observable (associated W_o is full rank), then we can place the eigenvalues of A – LC arbitrarily through appropriate choice of L.
- **Proof:** Note that the transpose of A – LC is $A^T - C^TL^T$ and in this form, this is the same as the eigenvalue placement problem for state space controllers.
- **Remark:** In MATLAB, use L= place(A', C', eigs)' to determine L

Comparison with Reachability & State Fdbk

- The pair (A,C) is observable ⇔ (A^T, C^T) is reachable
  - Proofs are similar
- Dynamics with state feedback K are (A – BK)
  Observer dynamics with gain L are (A – LC)
- The poles of (A – LC) can be chosen arbitrarily ⇔ The poles of ($A^T - C^TL^T$) can be chosen arbitrarily
  - Use place(A^T,C^T,λ)
  - Convergence depends on Re(λ)
- If (A,B) reachable, can transform to reachable canonical form.
  If (A,C) observable, can transform to observable canonical form.
  (Neither observability nor reachability depend on a change of variables)

Simple Example

Double-integrator: $\ddot{z} = u$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \\ y &= x_1 \end{aligned}$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$W_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Measure position, estimate full state (position and velocity)
- Simple guess:

$$\hat{x}_1 = y, \ \hat{x}_2 = \dot{y}$$

- Design observer:

$$\dot{\hat{x}} = A\hat{x} - L(y - C\hat{x}) + Bu$$

L=place(A',C',[-1;-1]) = [2 1];

$$A - LC = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$$

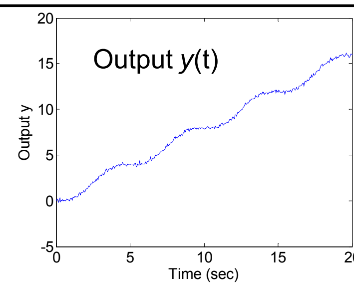
$$\begin{aligned} \dot{\hat{x}}_1 &= -2\hat{x}_1 + \hat{x}_2 + 2y \\ \dot{\hat{x}}_2 &= -\hat{x}_1 + y + u \end{aligned}$$

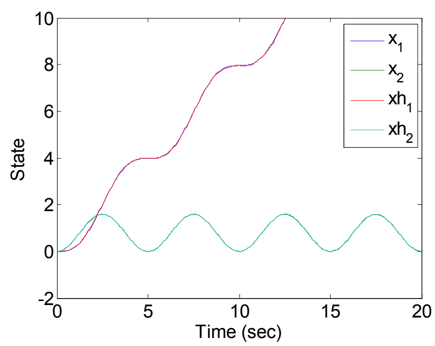
```
Wo=obsv(A,C);
L=place(A',C',[-1;-1.000001])';
sys=ss(A,B,C,0);
[u,t]=gensig('sin',5,20,dt);
[y,t,x]=lsim(sys,u,t);
y=y+0.1*randn(size(y));
% simple observer:
xh1=y;
xh2=diff([0;y])/dt;
% Luenberger observer:
Ao=A-L*C;
sysO=ss(Ao,[L B],eye(2),zeros(2,2));
xh=lsim(sysO,[y u],t);
```


Guess which observer is which?

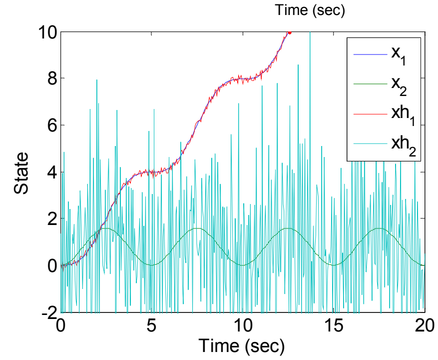
a) $\hat{x}_1 = y, \ \hat{x}_2 = \dot{y}$

b) $\dot{\hat{x}} = A\hat{x} - L(y - C\hat{x}) + Bu$



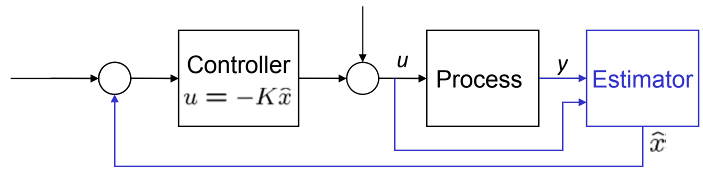


$$\dot{\hat{x}} = A\hat{x} - L(y - C\hat{x}) + Bu$$



$$\hat{x}_1 = y, \ \hat{x}_2 = \dot{y}$$

Separation Principle



- What happens when we apply state space controller using estimate of *x*?
  - We assumed we measured *x* directly in analyzing controller; extra dynamics in the estimator could cause closed loop to go unstable
- Thm: If K is a stabilizing compensator for (A, B) and L gives a stable estimator for (A, C), then the control law $u = -K\hat{x}$ is stable
  - This is an example of a separation principle: design the controller and estimator separately, then combine them and everything is OK
  - Be careful with signs on gains

Proof of Separation Principle

- Proof. Write down the dynamics for the complete system (assuming WLOG that x_d, u_d = 0):

$$\dot{x} = Ax + Bu \qquad \dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$
$$y = Cx \qquad u = -K\hat{x} + u_{\text{ref}}$$

- Rewrite in terms of the error dynamics $\tilde{x} = x - \hat{x}$ and combined state $x, \tilde{x}$:

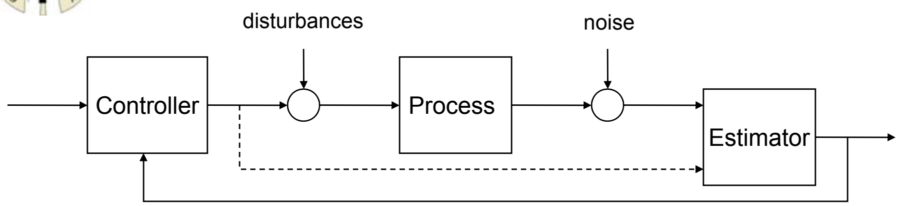
$$\dot{\tilde{x}} = (A - LC)\tilde{x} \qquad \frac{d}{dt}\begin{bmatrix} x \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}\begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} u_{\text{ref}} \\ 0 \end{bmatrix}$$

- Since the dynamics matrix is block diagonal, we find that the characteristic polynomial of the closed loop system is

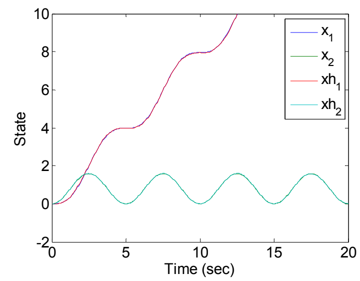
$$\det(sI - A + BK)\det(sI - A + LC).$$

- This polynomial is a product of two terms, where the first is the characteristic polynomial of the closed loop system obtained with state feedback and the other is the characteristic polynomial of the observer error.
- Since each was designed to be stable ⇒ the entire system is stable
  - Note that stability does not guarantee good performance!

Summary: Observers and State Estimation

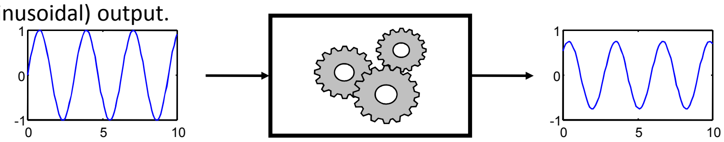


- Observability
  - Derived conditions for when we could determine state from inputs & outputs: check rank of observability matrix
- State Estimators
  - Construct state estimate based on prediction and correction (no noise yet)
- CDS112:
  - add noise to the problem $\Rightarrow$ Kalman filter
  - Minimize a quadratic performance $\Rightarrow$ LQR

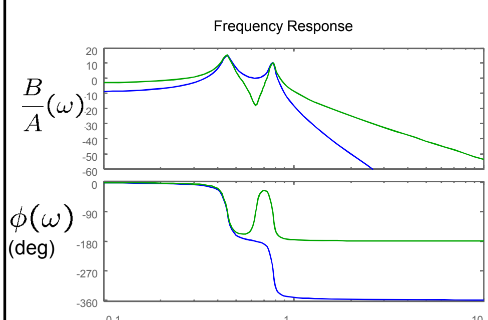


Frequency Domain Modeling (Ch 8)

- Definition: The frequency response of a linear system is the relationship between the gain and phase of a sinusoidal input and the corresponding steady state (sinusoidal) output.



$u = A\sin(\omega t)$ $\qquad$ $y = B\sin(\omega t + \phi)$

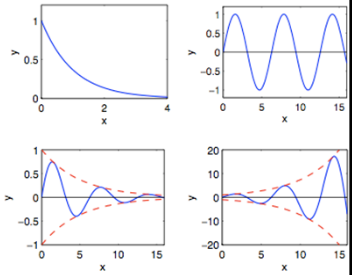


Bode plot (1940; Henrik Bode)

- Plot gain and phase vs input frequency
- Gain is plotting using log-log plot
- Phase is plotting with log-linear plot
- Can read off the system response to a sinusoid – in the lab or in simulations
- Linearity $\Rightarrow$ can construct response to any input (via Fourier decomposition)
- Key idea: do all computations in terms of gain and phase (frequency domain)

Transmission of Exponential Signals

- Exponential signal: $e^{st} = e^{(\sigma + i\omega)t} = e^{\sigma t} e^{i\omega t} = e^{\sigma t}(\cos \omega t + i \sin \omega t)$
  - Construct constant inputs + sines/cosines by linear combinations
    - Constant: $u(t) = c = ce^{0t}$
    - Sinusoid: $u(t) = A \sin(\omega t) = \frac{A}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$
  - Exponential response can be computed via the convolution equation



$$
\begin{aligned}
x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)} B e^{s\tau} \mathrm{d}\tau \\
&= e^{At}x(0) + e^{At}(sI - A)^{-1} e^{(sI-A)\tau}\Big|_{\tau=0}^{t} B \\
&= e^{At}x(0) + e^{At}(sI - A)^{-1}\left(e^{(sI-A)t} - I\right) B \\
&= e^{At}\left(x(0) - (sI - A)^{-1}B\right) + (sI - A)^{-1}Be^{st}
\end{aligned}
$$

$$
\begin{aligned}
y(t) &= Cx(t) + Du(t) \\
&= Ce^{At}\left(x(0) - (sI - A)^{-1}B\right) + \left(C(sI - A)^{-1}B + D\right)e^{st}
\end{aligned}
$$

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Transfer Function & Frequency Response

- Exponential response of a linear state space system

$$y = Ce^{At}\left(x(0) - (sI - A)^{-1}B\right) + \left(C(sI - A)^{-1}B + D\right)e^{st}$$

transient steady state

- Transfer function
  - Steady state response is proportional to exponential input => look at input/output

    $G(s) = C(sI - A)^{-1}B + D$ is the transfer function between input and output
- Frequency response

$$u(t) = A \sin \omega t = \frac{A}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$$

$$
\begin{aligned}
y_{ss}(t) &= \frac{A}{2i}\left(G(i\omega)e^{i\omega t} - G(-i\omega)e^{-i\omega t}\right) \\
&= A \cdot |G(i\omega)| \sin(\omega t + \arg G(i\omega))
\end{aligned}
$$

gain phase

Common transfer functions

$\dot{y} = u$	$\frac{1}{s}$
$y = \dot{u}$	s
$\dot{y} + ay = u$	$\frac{1}{s+a}$
$\ddot{y} = u$	$\frac{1}{s^2}$
$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = u$	$\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
$y = k_p u + k_d \dot{u} + k_i \int u$	$k_p + k_d s + \frac{k_i}{s}$
$y(t) = u(t - \tau)$	$e^{-\tau s}$

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