CDS 101/110: Lecture 3.2
Linear Systems, cont'd

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Goals:

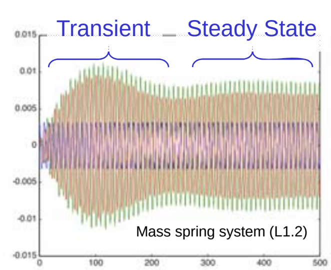
- Describe linear system response characteristics
  - Step response, impulse response
  - Frequency response
  - Jordan normal form (again!)

Reading:

- Åström and Murray, Analysis and Design of Feedback Systems, Ch 5

Input/Output Performance

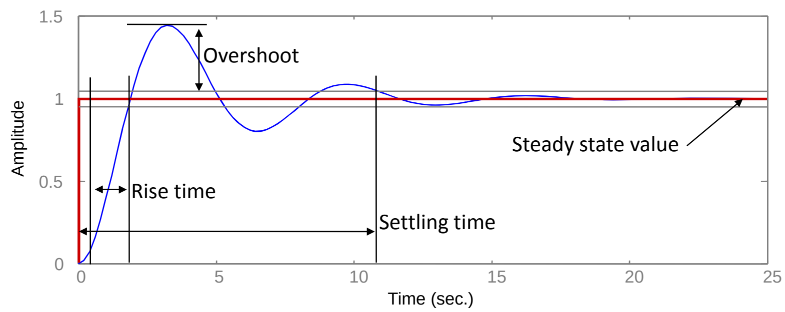
- How does system respond to changes in input values?
  - Transient response:
  - Steady state response:
- Characterize response in terms of
  - Impulse response
  - Step response
  - Frequency response



- Stability vs input/output performance
  - Systems that are close to instability typically exhibit poor input/output performance (slow convergence and/or "ringing" – a highly oscillatory response to [non-periodic] inputs)

Step Response

- Output characteristics in response to a "step" input
  - Rise time: time required to move from 5% to 95% of final value
  - Overshoot: ratio between amplitude of first peak and steady state value
  - Settling time: time required to remain w/in *p*% (usually 2%) of final value
  - Steady state value: final value at t = ∞



Second Order Systems

$$\ddot{q}+2\zeta\omega_0\dot{q}+\omega_0^2 q = u \qquad \leftrightarrow \qquad \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

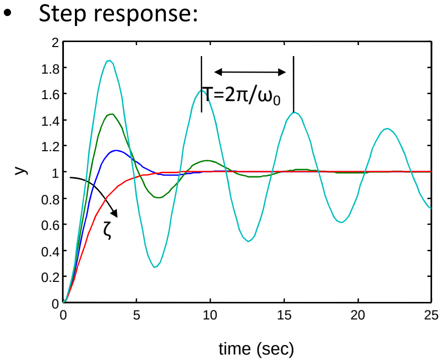
- Impulse response:

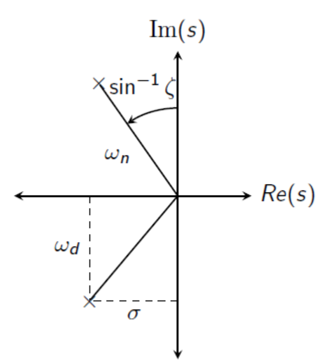
$$h(t) = \frac{\omega_0}{\sqrt{1-\zeta^2}} e^{-\sigma t} \sin \omega_d t$$

For $\zeta < 1$, eigenvalues at

$$s_{1,2} = \left(-\zeta \pm j\sqrt{1-\zeta^2}\right)\omega_0$$

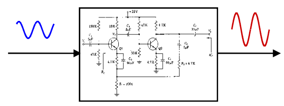
- Step response:



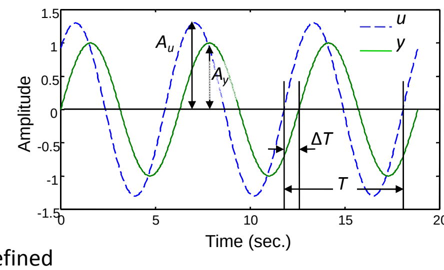


Frequency Response

- Measure the *steady state* response of the system to sinusoidal input
  - Example: audio amplifier – would like consistent ("flat") amplification between 20 Hz & 20,000 Hz
  - Individual sinusoids are good *test signals* for measuring performance in many systems (e.g., seasonal cycles in temperature)



- Approach: plot input and output, measure *relative* amplitude and phase
  - Use MATLAB or SIMULINK to generate response of system to sinusoidal output
  - Gain = A_y/A_u
  - Phase = $2\pi \cdot \Delta T/T$



- May not work for *nonlinear* systems
  - System nonlinearities can cause *harmonics* to appear in the output
  - Amplitude and phase may not be well-defined
  - For *linear* systems, frequency response is always well defined

Calculating Frequency Response from convolution equation

- Convolution equation describes response to any input; use this to look at response to sinusoidal input: $u(t) = A\sin(\omega t) = \frac{A}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$

u(t)

$$
\begin{aligned}
x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}B\boxed{e^{i\omega\tau}}\,\mathrm{d}\tau \\
&= e^{At}x(0) + e^{At}\int_0^t e^{(i\omega I - A)\tau}B\,\mathrm{d}\tau \\
&= e^{At}x(0) + e^{At}(i\omega I - A)^{-1}e^{(i\omega I - A)\tau}\Big|_{\tau=0}^{t}B \\
&= e^{At}x(0) + e^{At}(i\omega I - A)^{-1}\left(e^{(i\omega I - A)t} - I\right)B \\
&= \underbrace{e^{At}\left(x(0) - (i\omega I - A)^{-1}B\right)}_{\text{Transient (decays if stable)}} + \underbrace{(i\omega I - A)^{-1}B}_{\text{Ratio of response/input}}e^{i\omega t}
\end{aligned}
$$

$$
\begin{aligned}
y(t) &= Cx(t) + Du(t) \\
&= Ce^{At}\left(x(0) - (i\omega I - A)^{-1}B\right) + \boxed{\left(C(i\omega I - A)^{-1}B + D\right)}e^{i\omega t}
\end{aligned}
$$

"Frequency response"

Calculating Frequency Response from convolution equation #2 (more in 2 weeks)

- More generally, consider exponential windowing:

$$u(t) = e^{\sigma t} e^{i\omega t} = e^{(\sigma + i\omega)t} = e^{st}$$

u(t)

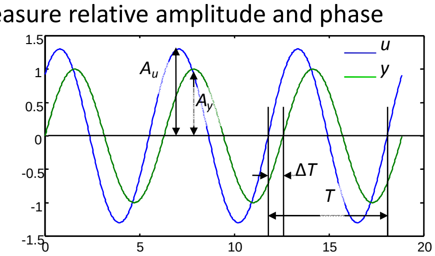
$$\begin{aligned} x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)} B e^{s\tau} d\tau \\ &= e^{At}x(0) + e^{At} \int_0^t e^{(sI-A)\tau} B d\tau \\ &= e^{At}x(0) + e^{At}(sI-A)^{-1} e^{(sI-A)\tau}\Big|_{\tau=0}^{t} B \\ &= e^{At}x(0) + e^{At}(sI-A)^{-1}\left(e^{(sI-A)t} - I\right) B \\ &= \underbrace{e^{At}\left(x(0) - (sI-A)^{-1}B\right)}_{\text{“Transient” (decays…)}} + \underbrace{(sI-A)^{-1}Be^{st}}_{\text{Ratio of response/input}} \end{aligned}$$

$$\begin{aligned} y(t) &= Cx(t) + Du(t) \\ &= Ce^{At}\left(x(0) - (sI-A)^{-1}B\right) + \boxed{\left(C(sI-A)^{-1}B + D\right)} e^{st} \end{aligned}$$

“Frequency response”

Computing Frequency Responses

- Technique #1: plot input and output, measure relative amplitude and phase
  - Generate response of system to sinusoidal output
  - Gain = A_y/A_u
  - Phase = $2\pi \cdot \Delta T/T$
  - For *linear* system, gain and phase don't depend on the input amplitude

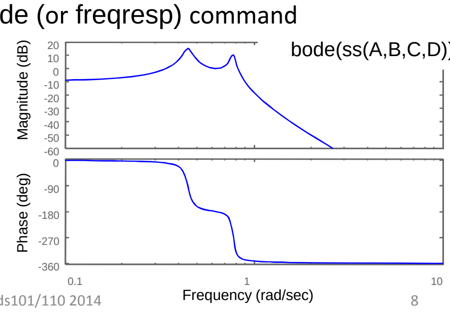


- Technique #2 (linear systems): use bode (or freqresp) command
  - Assumes linear dynamics in state space form:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

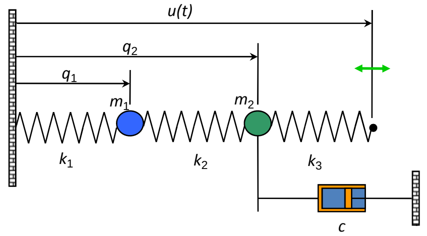
  - Gain plotted on log-log scale
    - dB = 20 $\log_{10}$ (gain)
  - Phase plotted on linear-log scale

bode(ss(A,B,C,D))



Spring Mass System

Frequency response: $C(j\omega I-A)^{-1}B+D$



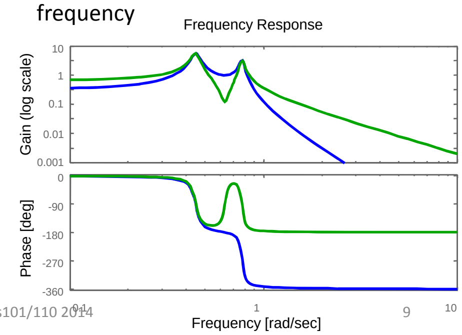
$$\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m} & \frac{k_2}{m} & 0 & 0 \\ \frac{k_2}{m} & -\frac{k_2+k_3}{m} & 0 & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

With $k_1 = k_2 = 1,\ m = 1,\ c = 0$

$$v_{1,2} = \begin{bmatrix} 1 \\ 1 \\ \pm 1i \\ \pm 1i \end{bmatrix} \qquad v_{3,4} = \begin{bmatrix} 1 \\ -1 \\ \pm\sqrt{2}i \\ \mp\sqrt{2}i \end{bmatrix}$$

Eigenvalues of A:

- For zero damping, $\pm j\omega_1$ and $\pm j\omega_2$
- ω_1 and ω_2 correspond to the two peaks in the frequency response
- The eigenvectors for these eigenvalues give the *mode shape*:
  - In-phase motion for the lower frequency
  - Out-of phase motion for the higher frequency



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