CDS101/110a: Introductory Control Theory
Lecture 3.1: Linear Systems

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Administrative information

- HW2 due Wed Oct 15
- TA office hours
  - Mon 3–5pm, 107 Annenberg
  - Tue **7–9pm**, 107 Annenberg
- Prof. MacMartin out today (no office hours)

today:

- linear time-invariant systems
- convolution equation
- step response, impulse response
- next time: frequency response

reading:

- Åström and Murray, Chapter 5

Linear systems

- many important examples: electronic circuits, mechanical systems, etc. are inherently linear
- even more systems are locally linear
  - often use feedback to make nonlinear system *seem* linear
  - frequency response is a key performance specification

why care? . . . because we have extremely good **tools**

- frequency and step response
- nyquist plots, gain/phase margin, loop shaping
- optimal control and estimators, LQR, Kalman filtering (CDS110b)
- robust control, H_2, H_∞, μ-analysis (CDS212)
- convex optimization, LMIs (CDS270 Sp15, shameless)

Part I: Autonomous systems

Autonomous system

- autonomous linear dynamical system

$$\begin{cases} \dot{x}(t) = Ax(t) \\ x(0) = x_0 \end{cases}$$

- **solution**: matrix exponential $x(t) = e^{At}x_0$

$$e^M \triangleq I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \cdots$$

- set $M = At$ and multiply out
  - in Matlab, `expm(..)` is matrix exponential
  - `exp(..)` is entrywise exponential

Properties of matrix exponential

for a matrix $M \in \mathbf{R}^{n \times n}$

$$e^M \stackrel{\Delta}{=} I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \cdots$$

- $e^M = \lim_{k \to \infty} (I + M/k)^k$
- if $n = 1$, becomes scalar exponential
- $e^{0_{n \times n}} = I_{n \times n}$
- $e^{M^T} = (e^M)^T$
- $\frac{d}{dt} e^{At} = Ae^{At} = e^{At}A$
- **false in general**: $e^{A+B} = e^A e^B$ (true only if $AB = BA$)

Computing the matrix exponential (diagonal)

suppose $M \in \mathbf{R}^{3\times 3}$ is diagonal:

$$M = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$M^2 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \cdot \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}$$

$$\vdots$$

$$M^k = \begin{bmatrix} \lambda_1^k & 0 & 0 \\ 0 & \lambda_2^k & 0 \\ 0 & 0 & \lambda_3^k \end{bmatrix}$$

Computing the matrix exponential (diagonal)

matrix exponential of a diagonal matrix:

$$
\begin{aligned}
e^M &= I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \cdots \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} + \cdots \\
&= \begin{bmatrix} 1 + \lambda_1 + \frac{\lambda_1^2}{2!} + \cdots & 0 & 0 \\ 0 & 1 + \lambda_2 + \frac{\lambda_2^2}{2!} + \cdots & 0 \\ 0 & 0 & 1 + \lambda_3 + \frac{\lambda_3^2}{2!} + \cdots \end{bmatrix} \\
&= \begin{bmatrix} e^{\lambda_1} & 0 & 0 \\ 0 & e^{\lambda_2} & 0 \\ 0 & 0 & e^{\lambda_3} \end{bmatrix}
\end{aligned}
$$

Computing the matrix exponential (diagonalizable)

suppose $M = VDV^{-1} \in \mathbf{R}^{n\times n}$, where

- $D \in \mathbf{R}^{n\times n}$ is diagonal
- $V \in \mathbf{R}^{n\times n}$ is invertible
- matlab: `[V,D] = eig(M)`

to calculate e^M we first note:

$$
\begin{aligned}
M^2 &= (VDV^{-1})(VDV^{-1}) = VD(V^{-1}V)DV^{-1} \\
&= VD^2V^{-1} \\
M^3 &= (VDV^{-1})(VDV^{-1})(VDV^{-1}) \\
&\vdots \\
M^k &= VD^kV^{-1}
\end{aligned}
$$

Computing the matrix exponential (diagonalizable)

for diagonalizable $M = VDV^{-1}$

$$\begin{aligned} e^M &= I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \cdots \\ &= I + (VDV^{-1}) + \frac{1}{2!}(VD^2V^{-1}) + \frac{1}{3!}(VD^3V^{-1}) + \cdots \\ &= V(I + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \cdots)V^{-1} \\ &= Ve^DV^{-1} \end{aligned}$$

Computing the matrix exponential (nilpotent)

suppose $N \in \mathbf{R}^{3\times 3}$ is a shift matrix:

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$N^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$N^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vdots$$

$$N^k = 0 \text{ for } k \geq 3$$

Computing the matrix exponential (nilpotent)

for a shift matrix $N \in \mathbf{R}^{3\times 3}$:

$$\begin{aligned} e^N &= I + N + \frac{1}{2!}N^2 + \frac{1}{3!}N^3 + \cdots \\ &= I + N + \frac{1}{2!}N^2 \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Computing the matrix exponential (general form)

- if M is not diagonalizable or nilpotent, we use the Jordan form
- roughly, we can always rewrite (interpretation of ÅM Th5.2)

$$M = \underbrace{VDV^{-1}}_{\text{diagonalizable}} + \underbrace{N}_{\text{nilpotent}},$$

$$\begin{aligned} e^M &= e^{VDV^{-1}+N} \\ &= e^{VDV^{-1}} \cdot e^N \quad \text{(matrices commute)} \\ &= Ve^D V^{-1} \cdot e^N \end{aligned}$$

- more next time...

Example

- find the solution $x(t)$ to

$$\begin{cases} \dot{x}(t) = Ax(t) \\ x(0) = x_0 \end{cases}$$

  with

$$A = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- **solution**: $x(t) = e^{At} x_0$

Example

- $M = At$ is real symmetric (diagonalizable)

$$A = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix} t = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^{-1} t$$

- matrix exponential is

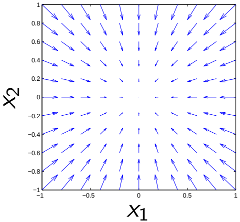
$$\begin{aligned} e^{At} &= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \exp \begin{bmatrix} 2t & 0 \\ 0 & t \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^{-1} \\ &= \frac{1}{2} \begin{bmatrix} e^{2t} + e^{t} & e^{2t} - e^{t} \\ e^{2t} - e^{t} & e^{2t} + e^{t} \end{bmatrix} \end{aligned}$$

- solution is

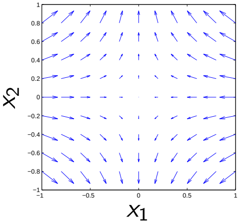
$$x(t) = \frac{1}{2} \begin{bmatrix} e^{2t} + e^{t} & e^{2t} - e^{t} \\ e^{2t} - e^{t} & e^{2t} + e^{t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{2t} \\ e^{2t} \end{bmatrix}$$

Eigenstructure of linear systems

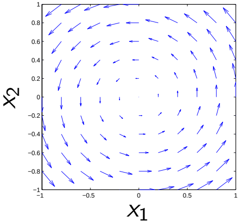
real evals, $Re(\lambda_1), Re(\lambda_2) < 0$:



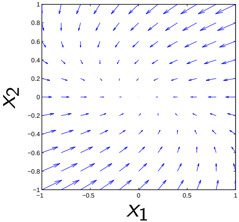
real evals, $Re(\lambda_1) < 0, Re(\lambda_2) > 0$:



cplx evals, $Re(\lambda_1) = Re(\lambda_2) = 0$:



cplx evals, $Re(\lambda_1), Re(\lambda_2) < 0$:



Stability of nonlinear systems

$$\begin{cases} \dot{x}(t) = f(x,u) \\ y(t) = h(x,u) \\ x(0) = x_0 \end{cases}$$

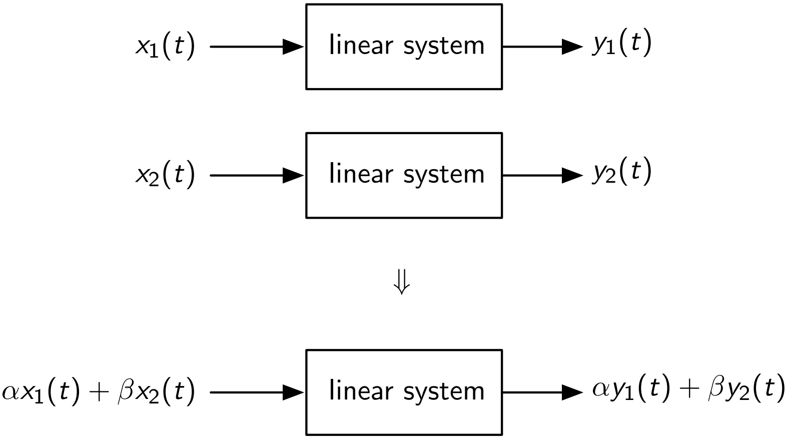
- assume equilibrium at (x_e, u_e), so that $f(x_e, u_e) = 0$
- change variables to $x - x_e$, $u - u_e$
- dynamics near equilibrium are approximately linear with

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x_e,u_e)} \quad B = \left.\frac{\partial f}{\partial u}\right|_{(x_e,u_e)}$$
$$C = \left.\frac{\partial h}{\partial x}\right|_{(x_e,u_e)} \quad D = \left.\frac{\partial h}{\partial u}\right|_{(x_e,u_e)}$$

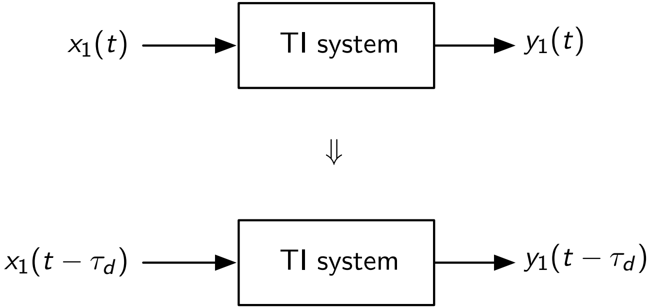
- local stability determined by eigenvalues of A
  caveat: if $Re\,\lambda_i(A) = 0$, need to go to higher order

Part II: Systems with inputs

Linearity


$x_1(t)$
linear system
$y_1(t)$
$x_2(t)$
linear system
$y_2(t)$
$\Downarrow$
$\alpha x_1(t) + \beta x_2(t)$
linear system
$\alpha y_1(t) + \beta y_2(t)$

Time invariance



Linear + Time Invariant (LTI) systems

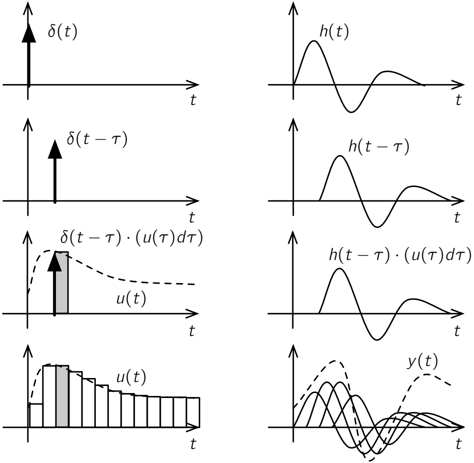
- **fact**: (ÅM §5.3) if a system is LTI, its output is a convolution

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)\, d\tau$$

- $u(t)$: input
- $y(t)$: output
- $h(t)$: impulse response fully characterizes system for any input

$$\begin{aligned} y(t) = (h * u)(t) &= \int_{-\infty}^{\infty} h(t-\tau)u(\tau)\ d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)u(t-\tau)\ d\tau \end{aligned}$$

Graphical interpretation



$$u(t) = \int_{-\infty}^{\infty} \delta(t-\tau)u(\tau)\ d\tau \qquad y(t) = \int_{-\infty}^{\infty} h(t-\tau)u(\tau)\ d\tau$$

System with inputs

- **fact**: any LTI system can be written in (A, B, C, D) form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \\ x(0) = x_0 \end{cases}$$

- **convolution**: assuming $u(t)$ is causal with $u(t) = 0$ for $t < 0$, the convolution equation for state is

$$x(t) = e^{At}x_0 + \int_{0^-}^{t} e^{A(t-\tau)}Bu(\tau)\, d\tau$$

- since $y(t) = Cx(t) + Du(t)$, the output is

$$y(t) = Ce^{At}x_0 + \int_{0^-}^{t} Ce^{A(t-\tau)}Bu(\tau)\, d\tau + Du(t)$$

State equation (proof)

$$x(t) = e^{At}x_0 + \int_{0^-}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$\Downarrow$ (differentiate under the integral) $\Downarrow$

$$\begin{aligned}
\dot{x}(t) &= \frac{d}{dt}\left\{e^{At}x_0 + \int_{0^-}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau\right\} \\
&= Ae^{At}x_0 + \int_{0^-}^{t} \frac{\partial}{\partial t}\left\{e^{A(t-\tau)}Bu(\tau)\right\}\,d\tau + \left. e^{A(t-\tau)}Bu(\tau)\right|_{\tau=t} \\
&= A\left(e^{At}x_0 + \int_{0^-}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau\right) + Bu(t) \\
&= Ax(t) + Bu(t) \quad \checkmark
\end{aligned}$$

Convolution equation

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \\ x(0) = x_0 \end{cases}$$

has solution

$$y(t) = \underbrace{Ce^{At}x_0}_{\text{unforced response}} + \underbrace{\int_{0^-}^{t} \underbrace{Ce^{A(t-\tau)}B}_{\text{impulse term}} \underbrace{u(\tau)}_{\text{input}} \, d\tau + \underbrace{Du(t)}_{\text{feedthrough}}}_{\text{forced response}}$$

Impulse response

suppose the input is an impulse $u(t) = \delta(t)$

$$\begin{aligned} y(t) &= Ce^{At}x_0 + \int_{0^-}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t) \\ &= Ce^{At}x_0 + \int_{0^-}^{t} Ce^{A(t-\tau)}B\delta(\tau)\,d\tau + D\delta(t) \\ &= Ce^{At}x_0 + Ce^{At}B + D\delta(t) \end{aligned}$$

- $Ce^{At}x_0$ is due to initial condition
- $h(t) = Ce^{At}B + D\delta(t)$ is the *impulse response*

Step response

suppose the input is a step $u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$

provided A is invertible, and for $t \geq 0$ we have

$$\begin{aligned} y(t) &= Ce^{At}x_0 + \int_{0^-}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t) \\ &= Ce^{At}x_0 + \int_{0^-}^{t} Ce^{A(t-\tau)}B\,d\tau + D \\ &= Ce^{At}x_0 + CA^{-1}(e^{At} - I)B + D \end{aligned}$$

- $Ce^{At}x_0$ is due to initial condition
- $s(t) = CA^{-1}(e^{At} - I)B + D$ is the *step response*
- step response is running integral $s(t) = \int_0^t h(\tau)\,d\tau$

Example

find the step response of

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \\ x(0) = x_0 \end{cases}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and the initial condition is $x_0 = 0$

Example

the matrix exponential e^{At} can be computed by diagonalizing

$$At = \begin{bmatrix} -j/\sqrt{2} & j/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} jt & 0 \\ 0 & -jt \end{bmatrix} \begin{bmatrix} j/\sqrt{2} & 1/\sqrt{2} \\ -j/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$\Downarrow$$

$$\begin{aligned} e^{At} &= \begin{bmatrix} -j/\sqrt{2} & j/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} e^{jt} & 0 \\ 0 & e^{-jt} \end{bmatrix} \begin{bmatrix} j/\sqrt{2} & 1/\sqrt{2} \\ -j/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \\ &= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \end{aligned}$$

Example

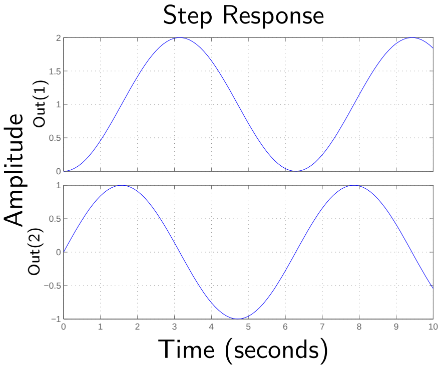
the step response is

$$\begin{aligned} s(t) &= CA^{-1}(e^{At} - I)B + D \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^{-1} \left(\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 - \cos t \\ \sin t \end{bmatrix} \end{aligned}$$

Simulating with Matlab

```
% state space system
A = [0 1; -1 0];
B = [0; 1];
C = eye(2);
D = [0; 0];
sys = ss(A,B,C,D);

% plot step responses
% up to 10 sec
step(sys, 10);
grid on;
```



other useful commands

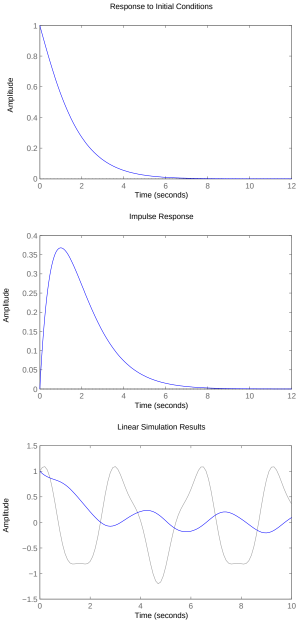
- `ss`, `tf`, `series`, `parallel`, `feedback` – define systems
- `gensig`, `square`, `sawtooth` – produce signals of diff. types
- `step`, `impulse`, `initial`, `lsim` – time domain analysis
- `bode`, `freqresp`, `evalfr` – frequency domain analysis
- `ltiview` – linear time invariant system plots

Another matlab example

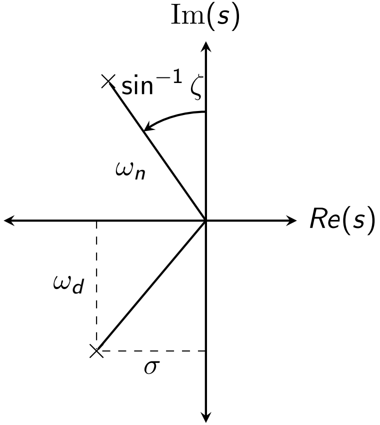
```
A = [-1 1; 0 -1]; B = [0; 1];
C = [1 0]; D = [0];
x0 = [1;0.5];

sys = ss(A,B,C,D);
initial(sys, x0);
impulse(sys);

t = 0:0.1:10;
u = 0.2*sin(5*t) + cos(2*t);
lsim(sys, u, t, x0);
```



Second-order system



- ODE with driving term $u(t)$

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \omega_n^2 u(t)$$

- impulse response, input $u(t) = \delta(t)$

$$h(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\sigma t} \sin(\omega_d t) \cdot 1(t)$$

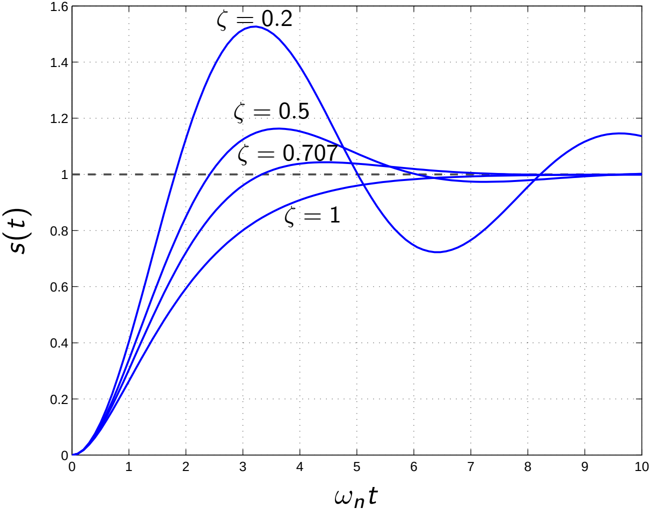
- step response, input $u(t) = 1(t) = \int_0^t \delta(\tau)\, d\tau$

$$s(t) = \int_0^t h(\tau)\, d\tau$$

- poles at $s = -\sigma \pm j\omega_d$

$$\sigma = \zeta\omega_n, \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

Second-order system: step response



Time domain specifications

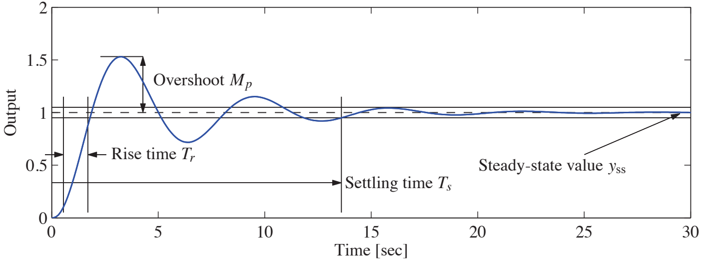


Figure 5.9: Sample step response. The rise time, overshoot, settling time and steady-state value give the key performance properties of the signal.

- rise time: time to move from 5% to 95% of final value
- overshoot: ratio between amplitude of first peak and ss value
- settling time: time required to remain within (*e.g.*) 2% of final value
- steady state value: final value as $t \to \infty$