CDS101/110a: Introductory Control Theory
Lecture 10.2: Design guidelines

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Administrative information

- HW8 (last one!) due Fri Dec 5
- will not be graded before final (plan accordingly)
- extra TA office hours Thu Dec 4 (7–9pm)

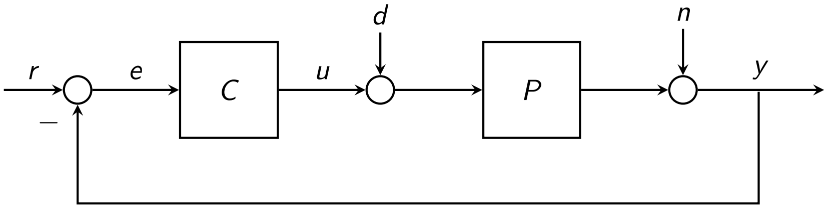
final:

- same style as midterm: 2 hours (CDS101), 3 hours (CDS110)
- open book/notes/course materials, closed internet
- due **8am Fri Dec 12**, hand to Nikki Fauntleroy (107 Steele)
- work clearly, efficiently: **we do not expect a perfect score**
- final review: Fri Dec 5

today:

- control design guidelines
- introduction to robust control

Gang of four



Loop transfer function $L(s) = P(s)C(s)$

$$H_{er}(s) = \frac{1}{1+L(s)} = S \qquad H_{yr}(s) = \frac{L(s)}{1+L(s)} = T$$

$$H_{yd}(s) = \frac{P(s)}{1+L(s)} = PS \qquad H_{un}(s) = \frac{C(s)}{1+L(s)} = CS$$

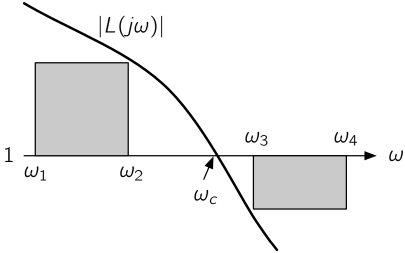
fact: $1 + L(s)$ is in the denominator

internal stability: all these must be stable

Design guidelines

open loop $L(s)$ determines performance and robustness

- $L(j\omega)$ large $\Rightarrow$ good "performance" at ω
- $L(j\omega)$ small $\Rightarrow$ good "robustness" at ω



Steady state error

typically:

- $|L(j\omega)| \gg 1$ at low frequency (tracking)
- $|L(j\omega)| \ll 1$ at high frequency (disturbance rejection)

steady state error to step input:

$$e_{ss} = \lim_{s \downarrow 0} \left[s \cdot \left(\frac{1}{1 + L(s)} \right) \left(\frac{1}{s} \right) \right] = \frac{1}{1 + L(0)}$$

- zero steady state error to step input requires an integrator
- zero steady state error to ramp input requires two integrators

Bode's gain-phase relationship

fact: for a stable minimum phase transfer function $L(s)$ with $L(0) > 0$ the phase at ω_0 is

$$\angle L(j\omega_0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d \ln |L(je^{\nu})|}{d\nu} \ln \coth \frac{|\nu|}{2} d\nu, \quad \nu = \ln(\omega/\omega_0).$$

exercise: can you show that

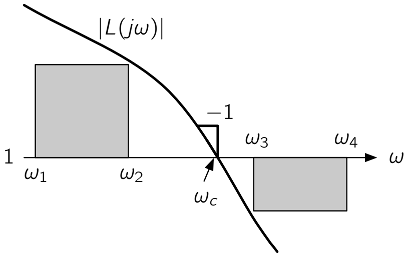
$$L(s) = \frac{1}{s^n} \quad \text{implies} \quad \angle L(j\omega) = -\frac{\pi}{2} n \text{ for all } \omega?$$

Bode's gain-phase relationship in practice

If $L(j\omega)$ locally "looks like" $\dfrac{1}{s^n}$, then $\angle L(j\omega) = -90^\circ n$.

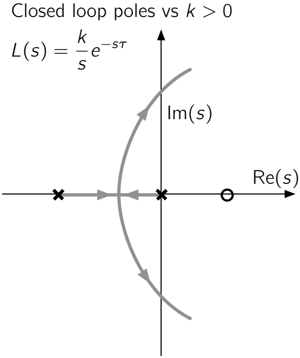
slope of $|L(j\omega_c)|$ **at crossover** ω_c:

- -2 slope of $|L|$ means no phase margin (bad)
- -1 slope of $|L|$ means 90° phase margin (good)
- 0 slope of $|L|$ means no crossover



Dominant poles

Closed loop performance typically close to second order system



- typically a dominant pair of poles that limit ability to increase loop gain further
- these are typically associated with phase margin at crossover

$$S(s) \simeq \frac{n(s)}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

where $\omega_n \sim$ loop crossover frequency

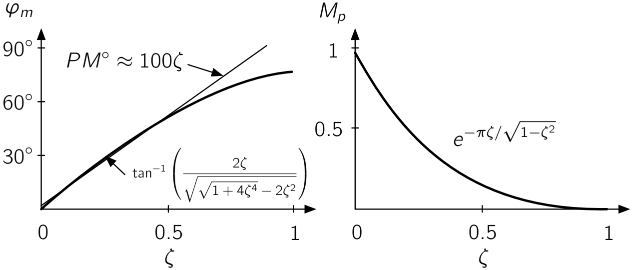
k	φ_m	ζ	ω_c	ω_n
0.5	62°	0.75	0.5	1
1	37°	0.35	1	1.4
1.5	16°	0.14	1.5	1.7

Phase margin

for the second order system

$$H(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

the phase margin and overshoot are



typically: we want $\varphi_m \geq 30^\circ$ for performance and robustness
(generally no advantage for $\varphi_m \geq 60^\circ$, *e.g.*, rise/settle time)

Alternative to working with L

- want $S(j\omega)$ small for good error tracking ($e = r - y$)
- Bode's integral formula precludes S small everywhere
- typically develop upper bound for $|S(j\omega)|$ and test

$$|S(j\omega)| < \frac{1}{|W(j\omega)|}, \quad \text{for all } \omega,$$

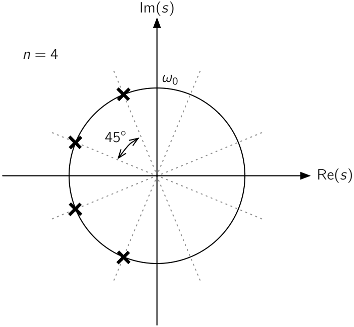
  where W is a weighting function
- equivalently

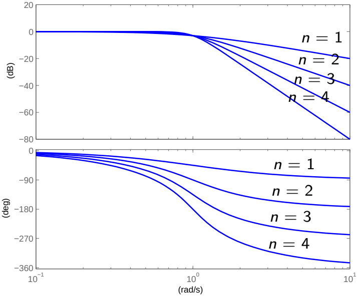
$$|W(j\omega)S(j\omega)| < 1, \quad \text{for all } \omega$$

Typical weighting functions

- leads and lags
- Butterworth filters: maximally "flat" passband with gentle rolloff into stopband

$$|W_n(j\omega)| = \frac{1}{\sqrt{1 + (\omega/\omega_0)^{2n}}}, \quad n = 1, 2, \ldots$$





Butterworth example

- lowpass at cutoff $\omega_0 = 1$ for $n = 2$

$$W_2(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

- highpass ($s \to 1/s$)

$$W_2(s) = \frac{s^2}{s^2 + \sqrt{2}s + 1}$$

- highpass at $\omega_0 \neq 1$ ($s \to s/\omega_0$)

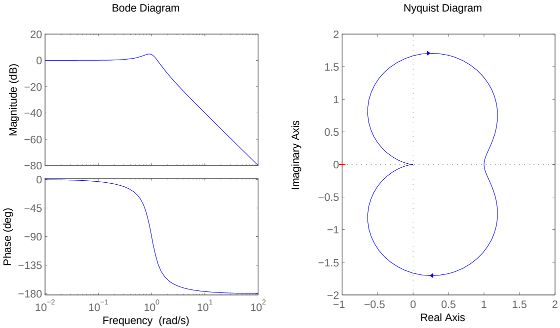
$$W_2(s) = \frac{(s/\omega_0)^2}{(s/\omega_0)^2 + \sqrt{2}(s/\omega_0) + 1}$$

Introduction to robust control

plant with uncertain damping

$$P(s) = \frac{1}{s^2 + as + 1}, \quad 0.4 \leq a \leq 0.8$$

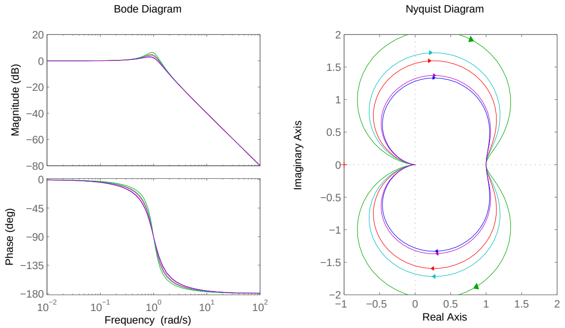
nominal plant has $a = a_0 = 0.6$



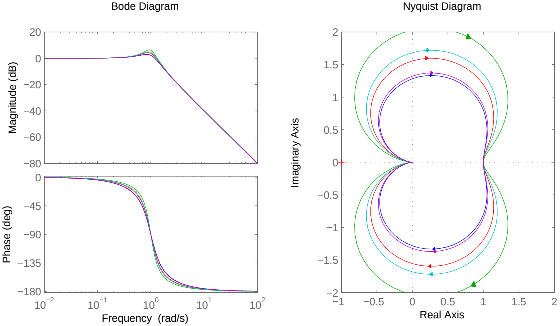
Robust stability

$$P(s) = \frac{1}{s^2 + as + 1}, \quad 0.4 \le a \le 0.8$$

- stability must hold for all values of $a \in [0.4, 0.8]$



Stability margins

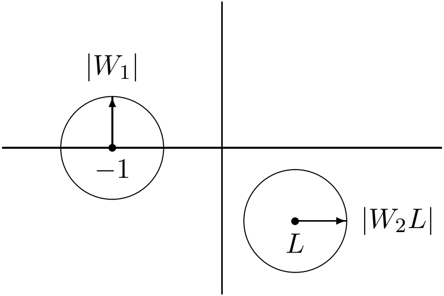


a	ζ	g_m	φ_m
0.4	0.2	∞	**32.8**°
0.5	0.25	∞	41.4°
$a_0 = 0.6$	0.3	∞	50.2°
0.7	0.35	∞	59.3°
0.8	0.4	∞	**68.9**°

Robust stability and performance on the Nyquist plot

fact: for a plant perturbation model $(1+\Delta W_2)P$ a necessary and sufficient condition for robust performance is

$$\max_{\omega} |W_1(j\omega)S(j\omega)| + |W_2(j\omega)T(j\omega)| < 1$$



more in CDS212–213

A taste of state space

Recall that the autonomous system

$$\dot{x}(t) = Ax(t), \quad x(t) \in \mathbf{R}^n,$$

is asymptotically stable if and only if

- all eigenvalues of $A \in \mathbf{R}^{n\times n}$ have negative real part
- there exists a quadratic Lyapunov function

$$\begin{aligned} V(x) &= x^T P x, \quad P = P^T \succ 0, \\ \dot{V}(x) &= x^T(A^T P + PA)x < 0 \quad \text{for all } x \neq 0 \end{aligned}$$

- the system of **linear matrix inequalities**

$$P \succ 0, \quad A^T P + PA \prec 0$$

  is feasible for some $P = P^T \in \mathbf{R}^{n\times n}$

Uncertain systems

Now consider the uncertain system (not LTI)

$$\dot{x}(t) \in \Omega x(t), \quad x(t) \in \mathbf{R}^n,$$

where Ω is a subset of $\mathbf{R}^{n\times n}$

- this is a differential inclusion (*cf.* differential equation)
- example sets

$$\begin{aligned}
\Omega &= \{A\}, \\
\Omega &= \{A_1, A_2, \ldots, A_L\}, \\
\Omega &= \{A + B\Delta C \mid \lambda_{\max}(\Delta^T \Delta) \le 1\}
\end{aligned}$$

Sample result: polytopic LDI

The linear differential inclusion

$$\dot{x}(t) \in \Omega x(t), \quad \Omega = \mathbf{conv}\{A_1, A_2, \ldots, A_L\}$$

has all trajectories converge to zero as $t \to \infty$ if there exists a joint Lyapunov function $V(x) = x^T P x$,

$$P \succ 0, \quad A_i^T P + P A_i \prec 0, \quad i = 1, \ldots, L$$

- a system of linear matrix inequalities in $P = P^T \in \mathbf{R}^{n \times n}$
- algorithms based on **linear algebra** and **convex optimization** can be used to find such a matrix
- *much more* in CDS270

Where to go from here

- **CDS112** Control System Design (Winter): optimal control/estimation, Kalman filtering, more state space
- **CDS140** Introduction to Dynamics (Winter): theory of linear/nonlinear ODEs, bifurcations, limit cycles
- **CDS212–213** Modern Control (Winter–Spring): even *more* state space, stability, realization theory, robust control, uncertainty modeling
- **CDS270** Advanced Topics (Spring): Lyapunov theory, optimization, LMIs, dissipative systems, applications