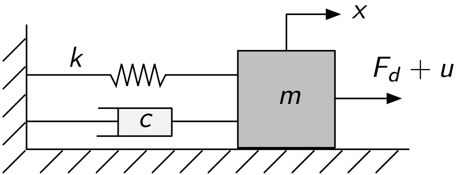
CDS101/110a: Introductory Control Theory

Recitation section
October 4, 2013

Administrative information

- website:
  - `http://www.cds.caltech.edu/~macmardg`
  - click on `CDS110a/101 (fa13)`
  - use piazza to ask questions
- email: `cds110-tas@cds.caltech.edu`
- office hours (check website):
  - Mon 3–5pm in 243 ANB
  - Tue 6–8pm in 328 SFL
- HW1 due Wed October 9, 5pm (box outside 102 Steele)
  - three 2-day grace periods
- (some) matlab today... more in-depth tutorial on Sunday

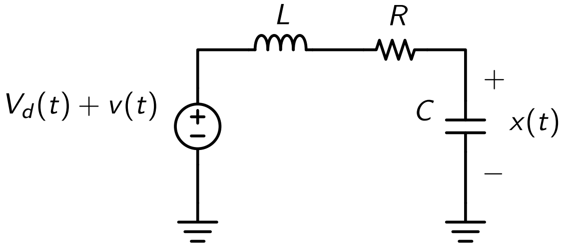
Introduction to feedback



$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F_d(t) + u(t)$$

- control input $u(t) \in \mathbf{R}$
- unknown disturbance $F_d(t) \in \mathbf{R}$
- initial conditions $x(0) = +1$ m and $\dot{x}(0) = 0$ m/s
- equilibrium at $x = 0$
- **goal**: choose u to stabilize around $x = 0$ despite disturbance

Linear system equivalence



$$LC\ddot{x}(t) + RC\dot{x}(t) + x(t) = V_d(t) + v(t)$$

(compare to)

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F_d(t) + u(t)$$

Solution to linear system

system is linear: if $x_1(t)$ and $x_2(t)$ each satisfy

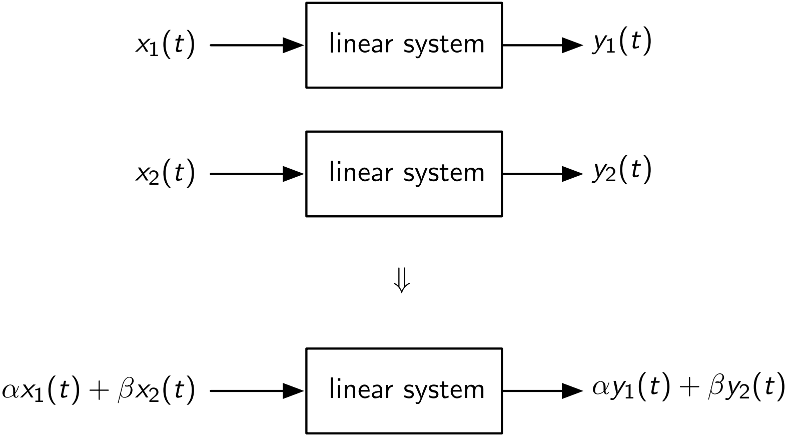
$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0,$$

then so does $z(t) = \alpha x_1(t) + \beta x_2(t)$ for any $\alpha, \beta \in \mathbf{R}$.

proof: derivative is a linear operator

$$\begin{aligned} m\ddot{z} + c\dot{z} + kz = \\ &= m(\alpha\ddot{x}_1 + \beta\ddot{x}_2) + c(\alpha\dot{x}_1 + \beta\dot{x}_2) + k(\alpha x_1 + \beta x_2) \\ &= \alpha(\underbrace{m\ddot{x}_1 + c\dot{x}_1 + kx_1}_{0}) + \beta(\underbrace{m\ddot{x}_2 + c\dot{x}_2 + kx_2}_{0}) \end{aligned}$$

Linearity


$x_1(t)$
linear system
$y_1(t)$
$x_2(t)$
linear system
$y_2(t)$
$\Downarrow$
$\alpha x_1(t) + \beta x_2(t)$
linear system
$\alpha y_1(t) + \beta y_2(t)$

State space formulation

- can also write

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}}_{A}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B} u(t) + \begin{bmatrix} 0 \\ \frac{F_d(t)}{m} \end{bmatrix}$$

- in general (with no disturbances)

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

- how do we compute $y(t)$ from $u(t)$?
- more next week...

Simulating in matlab

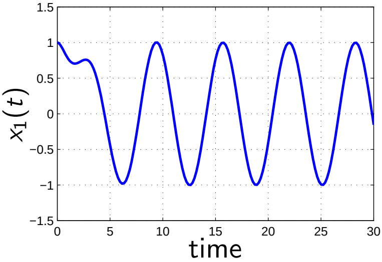
- suppose $F_d = 0$ and $u(t) = \sin(t)$

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$$

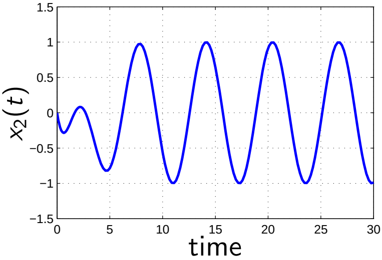
```
% system parameters
m = 1; k = 1; c = 1;
A = [0, 1; -k/m, -c/m];
B = [0; 1/m];
x0 = [1; 0];  % initial condition

% anonymous functions
u = @(t) sin(t);
odefun = @(t,x) A*x + B*u(t);

% simulate for 30 sec
[tout,xout] = ode45(odefun,[0,30],x0);
```

Plotting results

```
figure(1);
plot(tout, xout(:,1));
xlabel('time');
ylabel('x_1(t)');
ylim([-1.5,1.5]);
grid on;
```

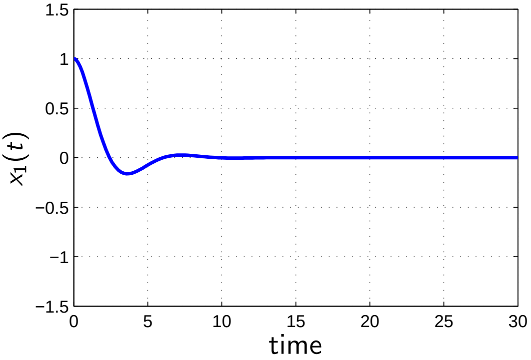



```
figure(2);
plot(tout, xout(:,2));
xlabel('time');
ylabel('x_2(t)');
ylim([-1.5,1.5]);
grid on;
```



Stability

- is system stable about $x = 0$ when $F_d(t) + u(t) = 0$?



...yes in simulation

Steady state values

- if system is stable, steady state implies no change

$$\dot{x} = 0, \quad \ddot{x} = 0$$

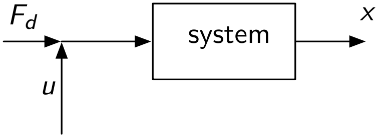
- steady state value x_{ss} is characterized by

$$\cancel{m\ddot{x}(t)} + \cancel{c\dot{x}(t)} + kx(t) = F_d(t) + u(t)$$

$$kx_{ss} = 0 \Rightarrow x_{ss} = 0$$

- if we could pick $u(t) = -F_d(t)$, guaranteed to be stable

Open loop control strategy



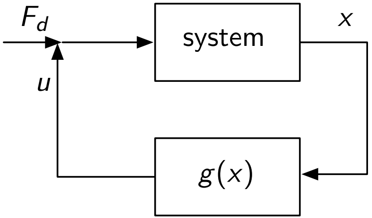
- suppose $F_d(t) = F_0$ and choose $u(t) = -\hat{F}_0$
- provided system is stable

$$x_{ss} = \frac{F_0 - \hat{F}_0}{k} \neq 0,$$

if our estimate $\hat{F}_0$ is inexact

- mitigated (somewhat) by stiff spring (k large)

Closed loop control strategy

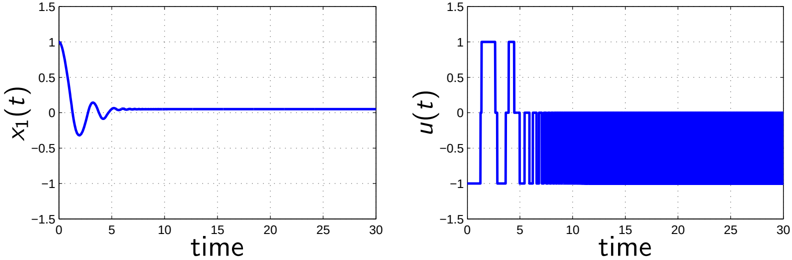


- choose input $u = g(x)$ based on state measurement

Saturated control

choose u = g(x) as "bang-bang" or "deadzone":

$$u = \begin{cases} u_{\max}, & x < 0 \\ 0, & x = 0 \\ -u_{\max}, & x > 0 \end{cases} \quad \text{or} \quad u = \begin{cases} u_{\max}, & x \leq -x_{\text{th}} \\ 0, & -x_{\text{th}} < x < x_{\text{th}} \\ -u_{\max}, & x \geq x_{\text{th}} \end{cases}$$



...difficult to analyze, but this is how your oven works

Proportional feedback

choose $u(t) = -k_p x(t)$:

- linear ($2\times$ error $\Rightarrow$ $2\times$ force)
- closed loop system is

$$m\ddot{x} + c\dot{x} + (k + k_p)x = F_d$$

- in steady state

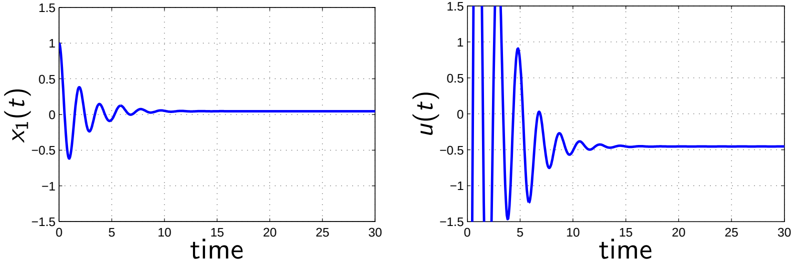
$$x_{ss} = \frac{F_d}{k + k_p}$$

- **benefit**: does not require knowledge of $F_d(t)$

Proportional feedback

here $F_d(t) = 0.5 \cdot 1(t-5)$ and $k_p = 10$

- transient: oscillations increased, damping unchanged
- steady state: good behavior if k_p is large (careful!)



PD (Proportional+Derivative) feedback

choose $u(t) = -k_p x(t) - k_d \dot{x}(t)$:

- closed loop system is

$$m\ddot{x} + (c + k_d)\dot{x} + (k + k_p)x = F_d$$

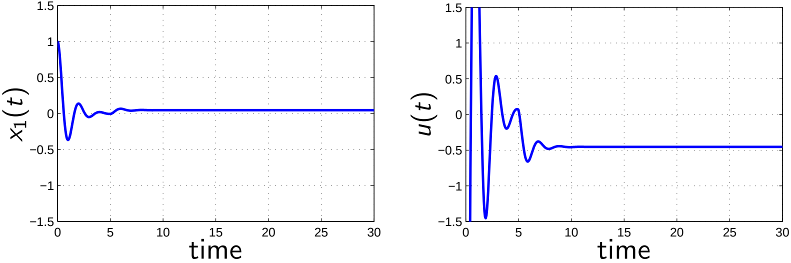
- in steady state

$$x_{ss} = \frac{F_d}{k + k_p}$$

PD (Proportional+Derivative) feedback

here $F_d(t) = 0.5 \cdot 1(t-5)$ and $k_p = 10$, $k_d = 1$

- transient: damping increased
- steady state: still have nonzero ss error



PI (Proportional+Integral) feedback

choose $u(t) = -k_p x(t) - k_i \int_0^t x(\tau)\, d\tau$:

- augment the state by letting

$$q = \int_0^t x(\tau)\, d\tau \quad \Rightarrow \quad \dot{q} = x(t)$$

- in state space form

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \\ q \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -\frac{k}{m} & -\frac{c}{m} & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ \frac{F_d(t)}{m} \\ 0 \end{bmatrix}$$

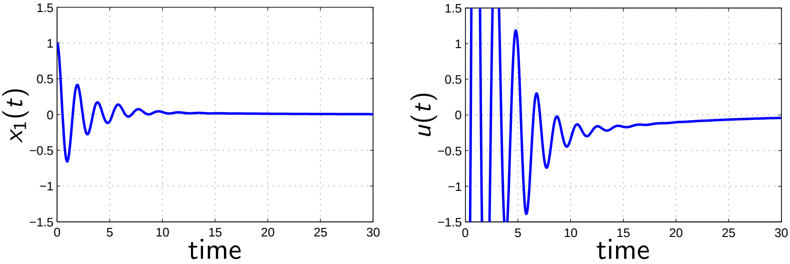
- **integral action**: if system is stable

$$\dot{q} = 0 \Rightarrow x(t) = 0$$

PI (Proportional+Integral) feedback

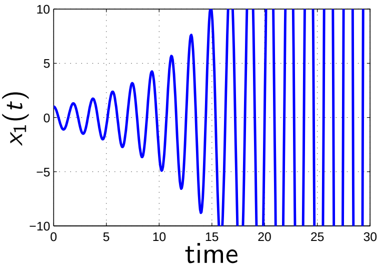
here $F_d(t) = 0.5 \cdot 1(t-5)$ and $k_p = 10$, $k_i = 1$

- transient: lack of D term to give extra damping
- steady state: ability to reject constant disturbance



What can go wrong?

PI controller with $F_d(t) = 0.5 \cdot 1(t-5)$ and $k_p = 10$, $k_i = 15$



...feedback destabilizes system

Goals for control theory

- characterize stability
- characterize (linear) system response
  - time domain: given $u(t)$ what is $x(t)$?
  - frequency domain: given $u(t) = \sin(\omega t)$, what is $x(t) = A(\omega)\sin(\omega t + \phi(\omega))$?
    (or given $u = e^{st}$, $x(t) = H(s)e^{st}$)
- how to choose control gains (*e.g.*, k_p, k_d, k_i) for
  - stability
  - disturbance rejection
  - track desired "command" (reference)
  - modify system dynamics
  - robustness: maintain stability despite modeling errors
- what are the "optimal" control gains?