# CDS 101/110a: Lecture 1.2
# System Modeling

**Douglas G. MacMartin**

**Goals:**
- Define a "model" and its use in answering questions about a system
- Introduce the concepts of state, dynamics, inputs and outputs
- Review modeling using ordinary differential equations (ODEs)

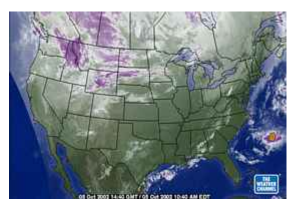
**Reading:**
- Åström and Murray, *Feedback Systems*, Sections 2.1–2.3, [40 min]
- Advanced: Lewis, *A Mathematical Approach to Classical Control*, Ch. 1



# Model-Based Analysis of Feedback Systems

- Analysis and design based on models
  - A model provides a prediction of how the system will behave
  - Feedback can give counter-intuitive behavior; models help sort out what is going on
  - For control design, models don't have to be exact: feedback provides robustness
- The model you use depends on the questions you want to answer
  - A single system may have many models
  - Time and spatial scale must be chosen to suit the questions you want to answer
  - Formulate questions before building a model
- Control-oriented models: inputs and outputs
  - Capture input/output behaviour "sufficiently" well

**Weather Forecasting**

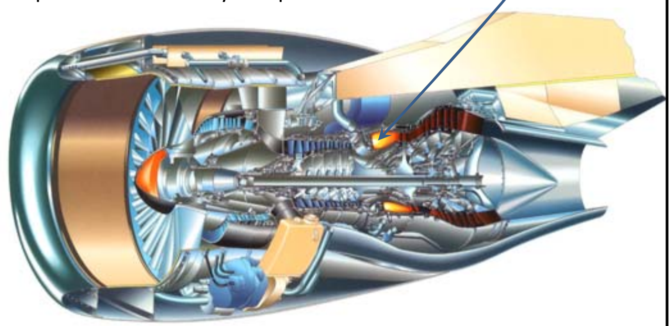


- **Question 1: how much will it rain tomorrow?**
- **Question 2: will it rain in the next 5-10 days?**
- **Question 3: will we have a drought next summer?**

***Different questions lead to different models***

# Example #0: Combustion Instability

- Thermoacoustic coupling in combustion chamber
- Approach: Modulate fuel supply (rapidly) in response to measured pressure oscillations
- "Standard" combustion model:
  - High-resolution coupled fluid-chemistry computational model
- "Controls" model:
  - Input/output
  - Simple block-diagram model that describes macroscopic heat release and coupling to acoustics…



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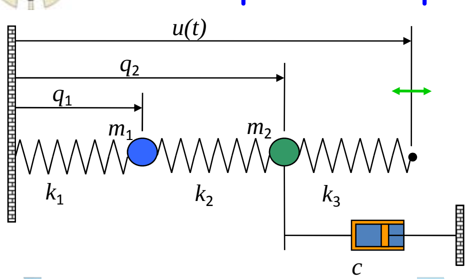
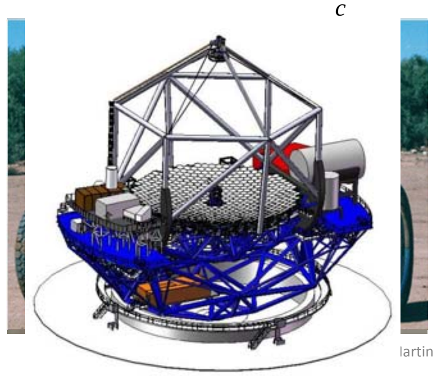
# Example #1: Spring Mass System



- Applications
  - Flexible structures (many apps)
  - Suspension systems (e.g., "Bob")
  - Molecular and quantum dynamics
- Questions we want to answer
  - How much do masses move as a function of the forcing frequency?
  - What happens if I change the values of the masses?
  - Will Bob fly into the air if I take that speed bump at 25 mph?
- Modeling assumptions
  - Mass, spring, and damper constants are fixed and known
  - Springs satisfy Hooke's law
  - Damper is (linear) viscous force, proportional to velocity

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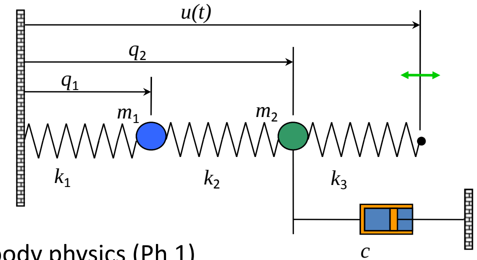
# Modeling a Spring Mass System




- Model: rigid body physics (Ph 1)
  - Sum of forces = mass * acceleration
  - Hooke's law: $F = k(x - x_{rest})$
  - Viscous friction: $F = c\,v$

$$m_1\ddot{q}_1 = k_2(q_2 - q_1) - k_1 q_1$$
$$m_2\ddot{q}_2 = k_3(u - q_2) - k_2(q_2 - q_1) - c\dot{q}_2$$

Can always re-write in first-order form:

$$\dot{x} = f(x,u) \qquad y = h(x)$$

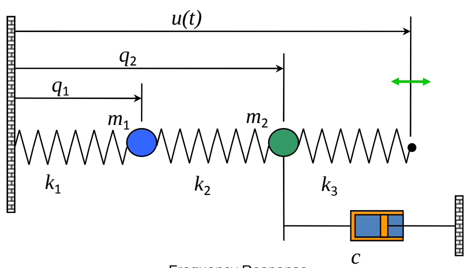
$$\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \frac{k_2}{m}(q_2 - q_1) - \frac{k_1}{m}q_1 \\ \frac{k_3}{m}(u - q_2) - \frac{k_2}{m}(q_2 - q_1) - \frac{c}{m}\dot{q}_2 \end{bmatrix}$$

$$y = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

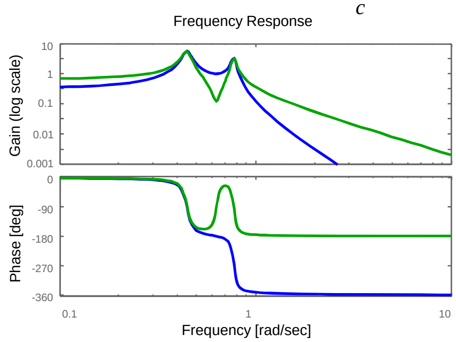
"State space form"



# Simulation of a Mass Spring System




- Steady state frequency response
  - Force the system with a sinusoid
  - Plot the "steady state" response, after transients have died out
  - Plot relative magnitude and phase of output versus input (more later)

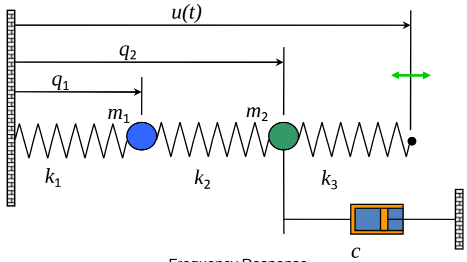


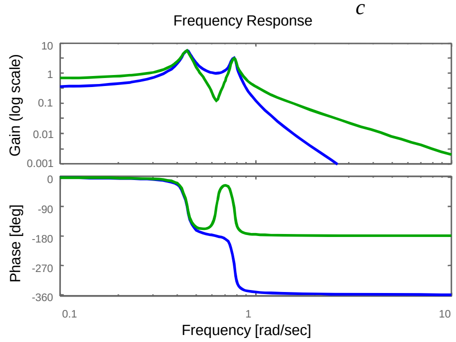

Matlab simulation (see handout)

```
function dydt = f(t, y, ...)
u = 0.00315*cos(omega*t);
dydt = [
  y(3);
  y(4);
  -(k1+k2)/m1*y(1) + k2/m1*y(2);
  k2/m2*y(1) - (k2+k3)/m2*y(2)
      - c/m2*y(4) + k3/m2*u ];

[t,y] = ode45(dydt,tspan,y0,[],
k1, k2, k3, m1, m2, c, omega);
```

## Simulation of a Mass Spring System




- Prelude… (we will revisit this in week 3)
  - System resonances are described by eigenvalues and eigenvectors of "A" matrix

$$\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \frac{k_2}{m}(q_2 - q_1) - \frac{k_1}{m}q_1 \\ \frac{k_3}{m}(u - q_2) - \frac{k_2}{m}(q_2 - q_1) - \frac{c}{m}\dot{q}_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \star & \star & 0 & 0 \\ \star & \star & 0 & \star \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ k_3/m \end{bmatrix} u$$

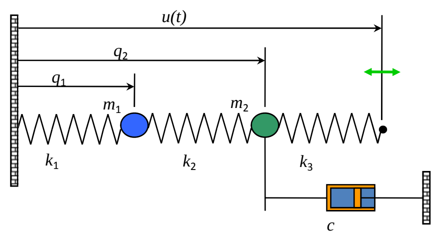
$$= Ax + Bu$$



## Modeling Terminology

$$\dot{x} = f(x, u)$$
$$y = h(x, u)$$

- State captures effects of the past
  - Independent physical quantities that determines future evolution (absent external excitation)
- Inputs describe external excitation
  - Inputs are extrinsic to the system dynamics (externally specified)
  - Disturbances & control inputs
- Dynamics describes state evolution
  - Update rule for system state
  - Function of current state and any external inputs
- Outputs describe measured quantities
  - Outputs are function of state and inputs; not independent variables
  - Outputs are often subset of state




**Example: spring mass system**

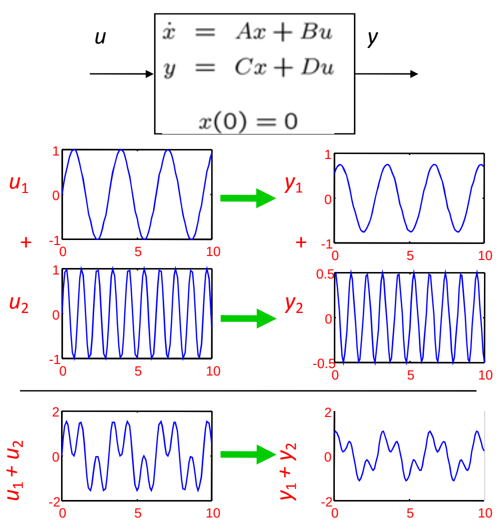
- State: position and velocity of each mass: $q_1,\ q_2,\ \dot{q}_1,\ \dot{q}_2$
- Input: position of spring at right end of chain: $u(t)$
- Dynamics: basic mechanics
- Output: measured positions of the masses: $q_1,\ q_2$

# Modeling Properties

- Choice of state is not unique
  - There may be many choices of variables that can act as the state
  - Trivial example: different choices of units (scaling factor)
  - Slightly less trivial example: sums and differences of the mass positions
- Choice of inputs and outputs depends on point of view
  - Inputs: what factors are external to the model that you are building
  - Inputs in one model might be outputs of another model (e.g., the output of a cruise controller provides the input to the vehicle model)
  - Outputs: what physical variables (often states) can you measure
  - Choice of outputs depends on what you can sense and what parts of the component model interact with other component models
- Can also have different types of models
  - Ordinary differential equations for rigid body mechanics
  - Difference equations
  - Finite state machines for manufacturing, Internet, information flow
  - Partial differential equations for fluid flow, solid mechanics, etc.



# Linear Systems




- A system is *linear* if the response is linear in initial condition & inputs:
  - $y(t)$ obtained with $x(0)=x_0$ and $u(t)=u_1(t)+u_2(t)$
  - Equals $y(t)$ due to
    - $x(0)=x_0$ (with $u(t)=0$) plus
    - $y(t)$ due to $u_1(t)$ (with $x(0)=0$) plus
    - $y(t)$ due to $u_2(t)$ (with $x(0)=0$)
- General form for *nonlinear* system:

$$\dot{x} = f(x,u)$$
$$y = h(x,u)$$

- General form for *linear* system:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$
$$x \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n \times n}$$

# State Transformation

- More generally, given
$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx + Du\end{aligned}$$
- And an *invertible* state transformation: (if $x \in \mathbb{R}^n, \quad V \in \mathbb{R}^{n\times n}$ )
$$z = Vx \qquad x = V^{-1}z$$
- Then the system can also be written as
$$\begin{aligned}\dot{z} &= (VAV^{-1})z + (VB)u\\ y &= (CV^{-1})z + Du\end{aligned}$$
- Which has the same input-output properties, though a different state vector



# More General Forms of Differential Equations

$$\frac{dx}{dt} = f(x,u) \qquad y = h(x,u)$$
General form

$$\frac{dx}{dt} = Ax + Bu \qquad y = Cx + Du$$
Linear system

$x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$
$y \in \mathbb{R}^q$

*x* = state; $n^{th}$ order
*u* = input; in 101/110a, usually p = 1
*y* = output; in 101/110a, usually q = 1

$$\frac{d^n q}{dt^n} + a_1 \frac{d^{n-1}q}{dt^{n-1}} + \cdots + a_n q = u$$
$$y = b_1 \frac{d^{n-1}q}{dt^{n-1}} + \cdots + b_{n-1}\dot{q} + b_n q$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} d^{n-1}q/dt^{n-1} \\ d^{n-2}q/dt^{n-2} \\ \vdots \\ dq/dt \\ q \end{bmatrix}$$

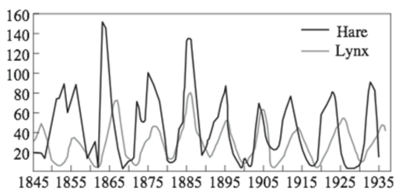
$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} x$$

# Difference Equations

- Difference equations model discrete transitions between continuous variables
  - "Discrete time" description (clocked transitions)
  - New state is function of current state + inputs
  - State is represented as a *continuous* variable

$$x[k+1] = f(x[k], u[k])$$
$$y[k] = h(x[k])$$

Example: predator prey dynamics




**Questions we want to answer**

- Given the current population of hares and lynxes, what will it be next year?
- If we hunt down lots of lynx in a given year, how will the populations be affected?
- How do long term changes in the amount of food available affect the populations?

**Modeling assumptions**

- Track population annual (discrete time)
- The predator species is totally dependent on the prey species as its only food supply
- The prey species has an external food supply and no threat to its growth other than the specific predator.



# Example #2: Predator Prey Modeling

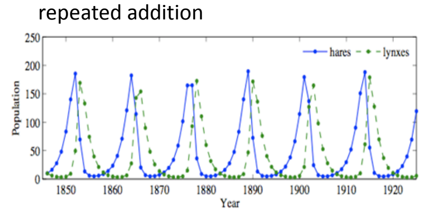
- Discrete Lotka-Volterra model
  - State
    - $H[k]$ # of hares in period $k$
    - $L[k]$ # of lynx in period $k$
  - Inputs (optional)
    - $u[k]$ amount of hares' food
  - Outputs: # of hares and lynx
  - Dynamics: Lotka-Volterra eqs

$$H[k+1] = H[k] + b_r(u)H[k] - aL[k]H[k]$$
$$L[k+1] = L[k] + cL[k]H[k] - d_f L[k]$$

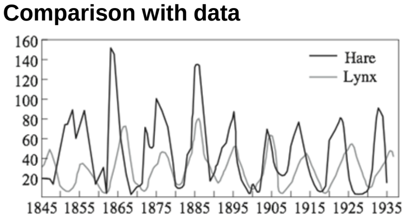
  - Parameters/functions
    - $b_r(u)$ hare birth rate (per period); depends on food supply
    - $d_f$ lynx mortality rate (per period)
    - $a$, $c$ interaction terms

**MATLAB simulation (see handout)**

- Discrete time model, "simulated" through repeated addition




**Comparison with data**



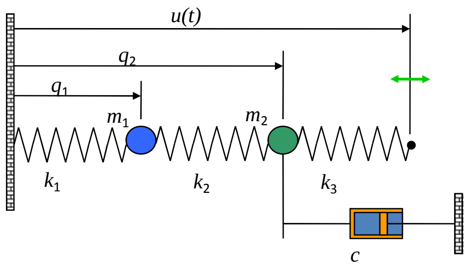
# Summary: System Modeling

- Model = state, inputs, outputs, dynamics

$$\frac{dx}{dt} = f(x,u)$$
$$y = h(x)$$

$$x[k+1] = f(x[k], u[k])$$
$$y[k] = h(x[k])$$

- *Principle*: Choice of model depends on the questions you want to answer




```
function dydt = f(t,y, k1, k2,
k3,   m1, m2, c, omega)
u = 0.00315*cos(omega*t);
dydt = [
  y(3);
  y(4);
  -(k1+k2)/m1*y(1) +
      k2/m1*y(2);
  k2/m2*y(1) - (k2+k3)/m2*y(2)
      - b/m2*y(4) + k3/m2*u ];
```

```
% L1_2_modeling.m - Lecture 1.2 MATLAB calculations
% RMM, 6 Oct 03

%
% Spring mass system
%

% Spring mass system parameters
m = 250; m1=m; m2=m;            % masses (all equal)
k = 50; k1=k; k2=k; k3=k;       % spring constants
b = 10;                 % damping
A = 0.00315; omega = 0.75;      % forcing function

% Call ode45 routine (MATLAB 6 format; help ode45 for details)
tspan=[0 500];              % time range for simulation
y0 = [0; 0; 0; 0];          % initial conditions
[t,y] = ode45(@springmass, tspan, y0, [], k1, k2, k3, m1, m2, b, A, omega);

% Plot the input and outputs over entire period
figure(1); clf
plot(t, A*cos(omega*t), t, y(:,1), t, y(:,2));

% Now plot the data for the final 10% (assuming this is long enough...)
endlen = round(length(t)/10);       % last 10% of data record
range = [length(t)-endlen:length(t)]';  % create vector of indices (note ')
tend = t(range);

figure(2); clf
plot(tend, A*cos(omega*tend), tend, y(range,1), tend, y(range,2));

% Compute the relative phase and amplitude of the signals
%
% We make use of the fact that we have a sinusoid in steady state,
% as well as its derivative.  This allows us to compute the magnitude
% of the sinusoid using simple trigonometry ( sin^2 + cos^2 = 1).

u = A*cos(omega*tend); udot = -A*omega*sin(omega*tend);
ampu = mean( sqrt((u .* u) + (udot/omega .* udot/omega)) );
fprintf(1, 'Amplitude = %0.5e cm', ampu*100);

%
% Predator prey system
%

% Set up the initial state
clear H L year
H(1) = 10; L(1) = 10;

% For simplicity, keep track of the year as well
year(1) = 1845;

% Set up parameters (note that c = a in the model below)
```

```
br = 0.6; df = 0.7; a = 0.014;
nperiods = 365;             % simulate each day
duration = 90;              % number of years for simulation

% Iterate the model
for k = 1:duration*nperiods
  b = br;                   % constant food supply
% b = br*(1+0.5*sin(2*pi*k/(4*nperiods)));  % varying food supply (try it!)
  H(k+1) = H(k) + (b*H(k) - a*L(k)*H(k))/nperiods;
  L(k+1) = L(k) + (a*L(k)*H(k) - df*L(k))/nperiods;
  year(k+1) = year(k) + 1/nperiods;

  if (mod(k, nperiods) == 1)
    % Store the annual population
    Ha((k-1)/nperiods + 1) = H(k);
    La((k-1)/nperiods + 1) = L(k);
  end;
end;

% Store the final population
Ha(duration) = H(duration*nperiods+1);
La(duration) = L(duration*nperiods+1);

% Plot the populations of rabbits and foxes versus time
figure(3); clf;
plot(1845 + [1:duration], Ha, '.-', 1845 + [1:duration], La, '.--');

% Adjust the parameters of the plot
axis([1845 1925 0 250]);
xlabel('Year');
ylabel('Population');

% Now reset the parameters to look like we want
lgh = legend(gca, 'hares', 'lynxes', 'Location', 'NorthEast', ...
  'Orientation', 'Horizontal');
legend(lgh, 'boxoff');
```

# Python code

```
# L1-3_modeling.py - Lecture 1.2 MATLAB calculations
# RMM, 23 Sep 2012
import numpy as np
import matplotlib.pyplot as mpl
from scipy.integrate import odeint

# Spring mass system
def springmass(y, t, A, omega):
    # Set the parameters
    k1 = 50.; k2 = 50.; k3 = 50. # spring constants
    m1 = 250.; m2 = 250.        # masses
    b = 10.                     # damping

    # compute the input to drive the system
    u = A * np.cos(omega*t)

    # compute the time derivative of the state vector
    dydt = (y[2], y[3],
            -(k1+k2)/m1*y[0] + k2/m1*y[1],
            k2/m2*y[0] - (k2+k3)/m2*y[1] - b/m2*y[3] + k3/m2*u)
    return dydt

# Call ode45 routine (MATLAB 6 format; help ode45 for details)
tspan = np.linspace(0, 500, 1000)     # time range for simulation
y0 = (0, 0, 0, 0);                    # initial conditions
A = 0.00315; omega = 0.75             # amplitude of forcing
sol = odeint(springmass, y0, tspan, (A, omega))
t = tspan

# Plot the input and outputs over entire period
mpl.figure(1); mpl.clf()
mpl.plot(t, A*np.cos(omega*t), t, sol[:,0], t, sol[:,1]);
mpl.show()
```

```
# Predator prey system
# Set up parameters (note that c = a in the model below)
br = 0.6; df = 0.7; a = 0.014;
nperiods = 365;                 # simulate each day
duration = 90;                          # number of years for simulation

# Set up the initial state
H = np.zeros(duration*nperiods); H[0] = 10;
L = np.zeros(duration*nperiods); L[0] = 10;

# For simplicity, keep track of the year as well
year = np.zeros(duration*nperiods); year[0] = 1845;

# Iterate the model
Ha = np.zeros(duration); La = np.zeros(duration);
for k in range(duration*nperiods-1):
    b = br;                                        # constant food supply
    # b = br*(1+0.5*sin(2*pi*k/(4*nperiods)));     # varying food supply (try it!)
    H[k+1] = H[k] + (b*H[k] - a*L[k]*H[k])/nperiods;
    L[k+1] = L[k] + (a*L[k]*H[k] - df*L[k])/nperiods;
    year[k+1] = year[k] + 1/nperiods;

    if (np.mod(k, nperiods) == 1):
        # Store the annual population
        Ha[k/nperiods] = H[k];
        La[k/nperiods] = L[k];

# Store the final population
Ha[duration-1] = H[duration*nperiods-1];
La[duration-1] = L[duration*nperiods-1];

mpl.plot(range(1845, 1845 + duration), Ha, '.-', \
      range(1845, 1845 + duration), La, '.--');
```