CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 101

D. MacMartin | **Problem Set #2** | Issued: 7 Oct 13
Fall 2013 | | Due: 16 Oct 13

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. Choose any *one* of the following systems, locate the equilibrium points for the system and indicate whether each is asymptotically stable, stable (but not asymptotically stable) or unstable. To determine stability, you can either use a phase portrait (if appropriate), analyze the linearization or simulate the system using multiple nearby initial conditions to determine how the state evolves.

   (a) *Nonlinear spring mass.* Consider a nonlinear spring mass system with dynamics

   $$m\ddot{q} = -k(q - aq^3) - c\dot{q},$$

   where $m = 1000$ kg is the mass, $k = 250$ kg/s^2 is the nominal spring constant, $a = 0.01$ represents the nonlinear "softening" coefficient of the spring and $c = 100$ kg/s is the damping coefficient. Note that this is very similar to the spring mass system we have studied in Section 2.2, except for the nonlinearity.

   (b) *Genetic toggle switch.* Consider the dynamics of two repressors connected together in a cycle. It can be shown (Exercise 2.9) that the normalized dynamics of the system can be written as

   $$\frac{dz_1}{d\tau} = \frac{\mu}{1 + z_2^n} - z_1 - v_1, \qquad \frac{dz_2}{d\tau} = \frac{\mu}{1 + z_1^n} - z_2 - v_2.$$

   where z_1 and z_2 represent scaled versions of the protein concentrations, v_1 and v_2 represent external inputs and the time scale has been changed. Let $\mu = 2.16$, $n = 2$ and $v_1 = v_2 = 0$.

   (c) *Congestion control of the Internet.* A simplified model for congestion control between N computers connected by a router is given by the differential equation

   $$\frac{dx_i}{dt} = -b\frac{x_i^2}{2} + (b_{\max} - b), \qquad \frac{db}{dt} = \left(\sum_{i=1}^{N} x_i\right) - c,$$

   where $x_i \in \mathbb{R}$, $i = 1, \ldots, N$ are the transmission rates for the sources of data, $b \in \mathbb{R}$ is the current buffer size of the router, $b_{\max} > 0$ is the maximum buffer size and $c > 0$ is the capacity of the link connecting the router to the computers. The $\dot{x}_i$ equation represents the control law that the individual computers use to determine how fast to send data across the network and the $\dot{b}$ equation represents the rate at which the buffer on the router fills up. Consider the case where $N = 2$ (so that we have three states, x_1, x_2 and b) and take $b_{\max} = 1$ Mb and $c = 2$ Mb/s.

2. Åström and Murray, Exercise 4.3 (Pay attention to the range over which you plot the phase portrait, e.g., from 15 to 25 m/s captures the "interesting" part of the velocity state.)

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CDS 110a

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Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. For *all* of the following systems, locate the equilibrium points for the system and indicate whether each is asymptotically stable, stable (but not asymptotically stable) or unstable. To determine stability, you can either use a phase portrait (if appropriate), analyze the linearization or simulate the system using multiple nearby initial conditions to determine how the state evolves.

   (a) *Nonlinear spring mass.* Consider a nonlinear spring mass system with dynamics

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2. Åström and Murray, Exercise 4.3 (Pay attention to the range over which you plot the phase portrait, e.g., from 15 to 25 m/s captures the "interesting" part of the velocity state.)

3. Åström and Murray, Exercise 4.4, changing second Lyapunov function to

$$V_2(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + \frac{b}{c-a}x_1)^2$$

(change in sign in second term if you are using the textbook; no change if you are using the online pdf).

4. Åström and Murray, Exercise 4.10