CDS 101/110a: Lecture 6-1
Transfer Functions

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Goals:

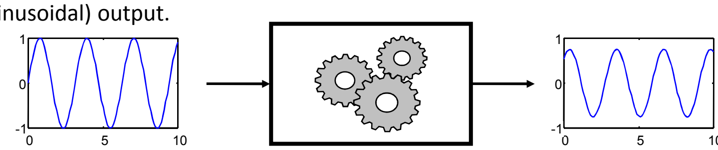
- Motivate and define the input/output transfer function of a linear system
- Understand the relationships among frequency response (Bode plot), transfer function, and state-space model
- Introduce block diagram algebra for transfer functions of interconnected systems

Reading:

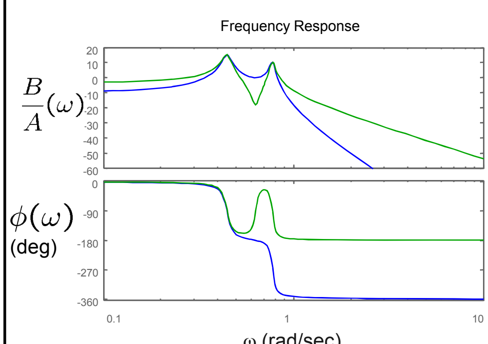
- Åström and Murray, *Feedback Systems,* Ch 8
- *Advanced*: Lewis, Chapters 3-4 or DFT, Chapter 2

Frequency Domain Modeling

Defn. The *frequency response* of a linear system is the relationship between the gain and phase of a sinusoidal input and the corresponding steady state (sinusoidal) output.



$u = A\sin(\omega t)$ $y = B\sin(\omega t + \phi)$

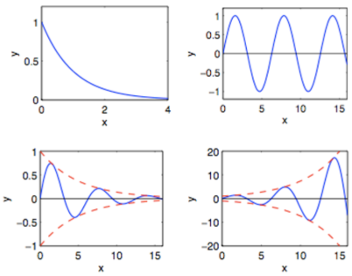


Bode plot (1940; Henrik Bode)

- Plot gain and phase vs input frequency
- Gain is plotting using log-log plot
- Phase is plotting with log-linear plot
- Can read off the system response to a sinusoid – in the lab or in simulations
- Linearity ⇒ can construct response to any input (via Fourier decomposition)
- Key idea: do all computations in terms of gain and phase (frequency domain)

Transmission of Exponential Signals

- Exponential signal: $e^{st} = e^{(\sigma+i\omega)t} = e^{\sigma t}e^{i\omega t} = e^{\sigma t}(\cos\omega t + i\sin\omega t)$
  - Construct constant inputs + sines/cosines by linear combinations
  - Constant: $u(t) = c = ce^{0t}$
  - Sinusoid: $u(t) = A\sin(\omega t) = \frac{A}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$



- Exponential response can be computed via the convolution equation

$$
\begin{aligned}
x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Be^{s\tau}\mathrm{d}\tau \\
&= e^{At}x(0) + e^{At}(sI-A)^{-1}e^{(sI-A)\tau}\Big|_{\tau=0}^{t} B \\
&= e^{At}x(0) + e^{At}(sI-A)^{-1}\left(e^{(sI-A)t} - I\right)B \\
&= e^{At}\left(x(0) - (sI-A)^{-1}B\right) + (sI-A)^{-1}Be^{st}
\end{aligned}
$$

$$
\begin{aligned}
y(t) &= Cx(t) + Du(t) \\
&= Ce^{At}\left(x(0) - (sI-A)^{-1}B\right) + \left(C(sI-A)^{-1}B + D\right)e^{st}
\end{aligned}
$$

- Can also motivate using Laplace transform: $Y(s)/U(s)=C(sI-A)^{-1}B+D$

Transfer Function and Frequency Response

- Exponential response of a linear state space system (from convolution)

$$y = Ce^{At}\left(x(0) - (sI-A)^{-1}B\right) + \left(C(sI-A)^{-1}B + D\right)e^{st}$$

transient — steady state

- Transfer function
  - Steady state response is proportional to exponential input => look at input/output ratio
  - $G(s) = C(sI-A)^{-1}B + D$ is the *transfer function* between input and output
- Frequency response

$$u(t) = A\sin\omega t = \frac{A}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$$

$$
\begin{aligned}
y_{ss}(t) &= \frac{A}{2i}\left(G(i\omega)e^{i\omega t} - G(-i\omega)e^{-i\omega t}\right) \\
&= A\cdot|G(i\omega)|\sin(\omega t + \arg G(i\omega))
\end{aligned}
$$

gain — phase

Common transfer functions

$\dot{y} = u$	$\frac{1}{s}$
$y = \dot{u}$	s
$\dot{y} + ay = u$	$\frac{1}{s+a}$
$\ddot{y} = u$	$\frac{1}{s^2}$
$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = u$	$\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
$y = k_p u + k_d\dot{u} + k_i\int u$	$k_p + k_d s + \frac{k_i}{s}$
$y(t) = u(t-\tau)$	$e^{-\tau s}$

Poles and Zeros

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$H(s) = \frac{n(s)}{d(s)}$$
$$d(s) = \det(sI - A)$$

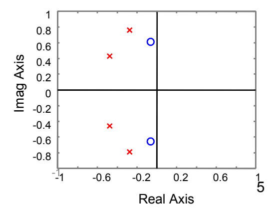
- Roots of $d(s)$ are called *poles* of $H(s)$
- Roots of $n(s)$ are called *zeros* of $H(s)$

- Poles of $H(s)$ determine the stability of the (closed loop) system
  - Denominator of transfer function = characteristic polynomial of state space system
  - Provides easy method for computing stability of systems
  - Right half plane (RHP) poles (Re > 0) correspond to unstable systems

- Zeros of $H(s)$ related to frequency ranges with limited transmission
  - A pure imaginary zero at $s=i\omega_z$ blocks any output at that frequency ($G(i\omega_z) = 0$)
  - Zeros provide limits on performance, especially RHP zeros (more on this later)

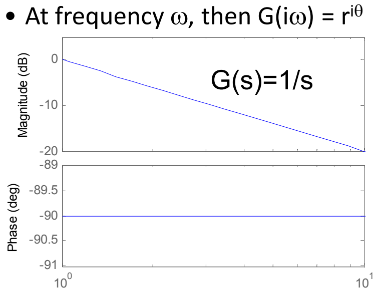
$$H(s) = k\frac{s^2 + b_1 s + b_2}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}$$

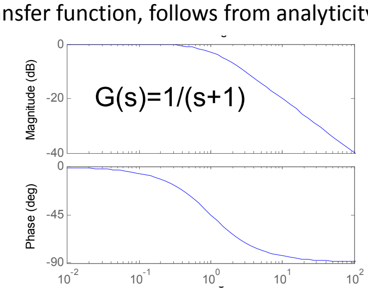
pzmap →

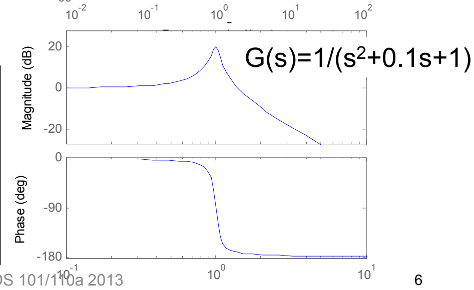


Plotting Bode Plots

- Evaluate the transfer function on the imaginary axis
  - This is sufficient to characterize the transfer function, follows from analyticity
- At frequency ω, then $G(i\omega) = r^{i\theta}$



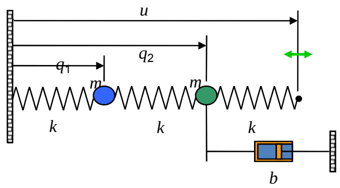




Some useful matlab commands:

```
sys=ss(A,B,C,D);
G=tf(sys);
G=ss2tf(A,B,C,D);
n=[0 0 1];d=[1 0.1 1],G=tf(n,d)
s=tf('s');G=1/(s^2+0.1*s+1);
bode(G)
```

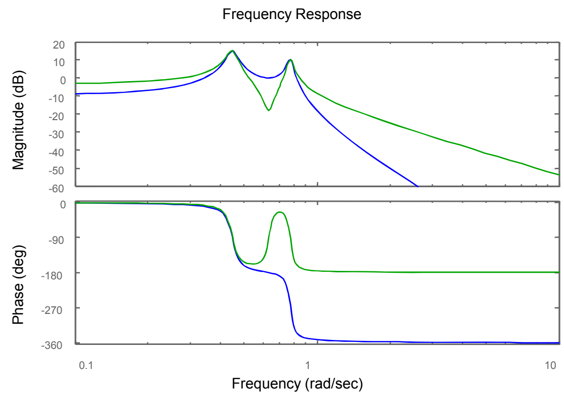
Example: Coupled Masses



$$H_{q_1 f}(s) = k\frac{0.04}{s^4 + 0.08s^3 + 0.8016s^2 + 0.032s + 0.12}$$

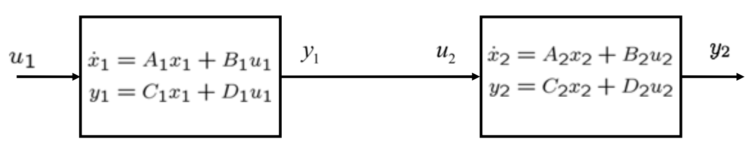
$$H_{q_2 f}(s) = k\frac{0.2s^2 + 0.008s + 0.08}{s^4 + 0.08s^3 + 0.8016s^2 + 0.032s + 0.12}$$

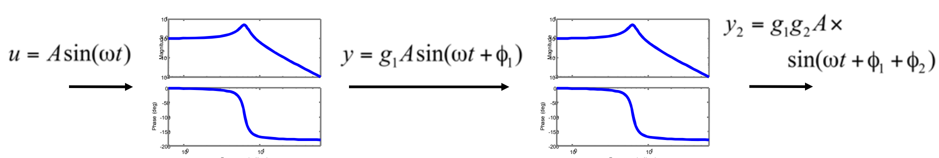
- Poles (H_{q1f} and H_{q2f})
  - -0.0200 ± 0.7743*j*
  - -0.0200 ± 0.4468*j*
- Zeros (H_{q2f})
  - -0.0200 ± 0.6321*j*
- Interpretation
  - Zeros in H_{q2f} give low response at $\omega \simeq 0.6321$



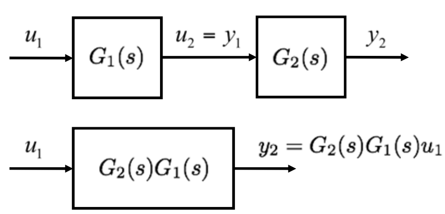
Series Interconnections

Q: what happens when we connect two systems together *in series?*

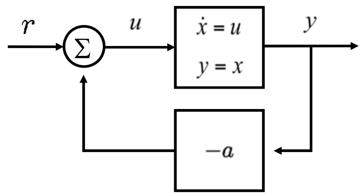


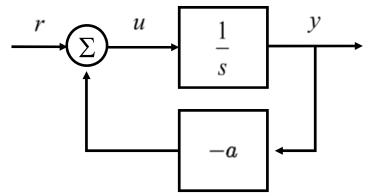


- A: Transfer functions *multiply*
  - Gains multiply
  - Phases add
  - Generally: transfer functions well formulated for frequency domain interconnections



Feedback Interconnection





- State space derivation

$$\dot{x} = u = r - ay = -ax + r$$
$$y = x$$

- Frequency response: $r = A\sin(\omega t)$

$$y = \left|\frac{1}{\sqrt{a^2+\omega^2}}\right| \sin\left(\omega t - \tan^{-1}\left(\frac{\omega}{a}\right)\right)$$

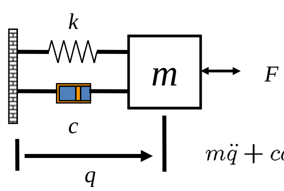
- Transfer function derivation

$$y = \frac{u}{s} = \frac{r - ay}{s}$$
$$y = \frac{r}{s+a} = G(s)r$$

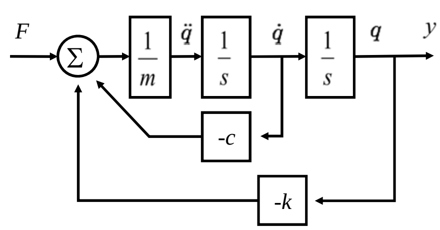
- Frequency response

$$y = |G(i\omega)| \sin(\omega t + \angle G(i\omega))$$

Example: mass spring system



$$m\ddot{q} + c\dot{q} + kq = F$$



- Rewrite in terms of "block diagram"
  - Represent integration using 1/*s*
  - Include spring and damping through feedback terms
  - Determine the transfer function through algebraic manipulation
  - Claim: resulting transfer function captures the frequency response

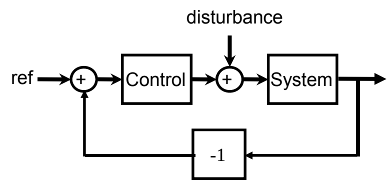
$$y = \frac{1}{m}\cdot\frac{1}{s}\cdot\frac{1}{s}(F - c\dot{q} - kq) = \frac{1}{ms^2}F - \frac{c}{ms}y - \frac{k}{ms^2}y$$

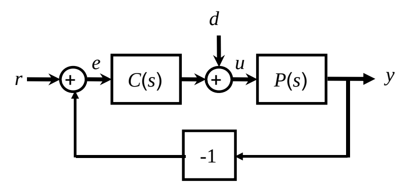
$$\left(1 + \frac{b}{ms} + \frac{k}{ms^2}\right) y = \frac{1}{ms^2}F$$

$$y = \frac{1}{ms^2 + cs + k}F$$

$$\boxed{H(s) = \frac{1}{ms^2 + cs + k}}$$

Control Analysis and Design Using Transfer Functions

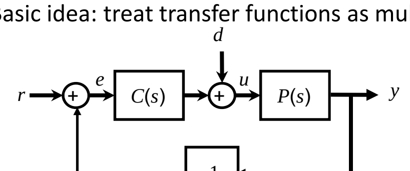




- Transfer functions provide a method for “block diagram algebra”
  - Easy to compute transfer functions between various inputs and outputs
    - $H_{er}(s)$ is the transfer function between the reference and the error
    - $H_{ed}(s)$ is the transfer function between the disturbance and the error
- Transfer functions provide a method for performance specification
  - Since transfer functions provide frequency response directly, it is convenient to work in the “frequency domain”
    - $H_{er}(s)$ should be small in the frequency range 0 to 10 Hz (good tracking)

Block Diagram Algebra

- Basic idea: treat transfer functions as multiplication, write down equations



$$y = P(s)u$$
$$u = d + C(s)e$$
$$e = r - y$$

- Manipulate equations to compute desired signals

$$\begin{aligned} e &= r - y \\ &= r - P(s)u \\ &= r - P(s)(d + C(s)e) \end{aligned}$$

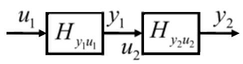
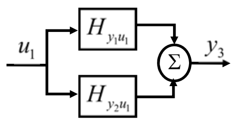
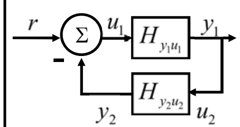
$$(1 + P(s)C(s))e = r - P(s)d$$

$$e = \underbrace{\frac{1}{1 + P(s)C(s)}}_{H_{er}} r - \underbrace{\frac{P(s)}{1 + P(s)C(s)}}_{H_{ed}} d$$

Note: linearity gives superposition of terms

- Algebra works because we are working in frequency domain
  - Time domain (ODE) representations are not as easy to work with
  - Formally, all of this works because of Laplace transforms

Block Diagram Algebra

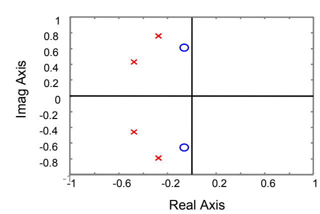
Type	Diagram	Transfer function
Series		$H_{y_2u_1} = H_{y_2u_2}H_{y_1u_1} = \frac{n_1n_2}{d_1d_2}$
Parallel		$H_{y_3u_1} = H_{y_2u_1} + H_{y_1u_1} = \frac{n_1d_2 + n_2d_1}{d_1d_2}$
Feedback		$H_{y_1r} = \frac{H_{y_1u_1}}{1 + H_{y_1u_1}H_{y_2u_2}} = \frac{n_1d_2}{n_1n_2 + d_1d_2}$

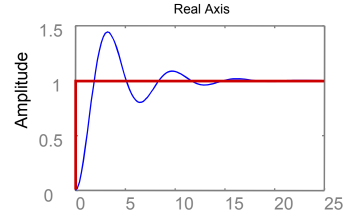
– These are the basic manipulations needed; some others are possible
– Formally, could work all of this out using the original ODEs (nothing *really* new)

MATLAB manipulation of transfer functions

- Creating transfer functions
  - [num, den] = ss2tf(A, B, C, D)
  - sys = tf(num, den) or tf(ss(A,B,C,D))
  - num=1, den = [1 a b] $\rightarrow s^2 + as + b$
- Interconnecting blocks
  - sys= series(sys1, sys2), parallel, feedback
- Computing poles and zeros
  - pole(sys), zero(sys)
  - pzmap(sys)
- I/O response
  - step(sys), bode(sys)

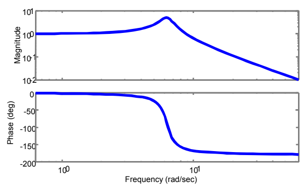
```
» tf(sys)
  Transfer function:
          1
   ---------------
   s^2 + 0.2 s + 1
```





Summary: Frequency Response & Transfer Functions

$u = A\sin(\omega t) \rightarrow$ $\dot{x} = Ax + Bu$, $y = Cx + Du$, $x(0) = 0$ $\rightarrow y_{ss} = A \cdot |G(i\omega)| \times \sin(\omega t + \arg G(i\omega))$



$$G(s) = C(sI - A)^{-1}B + D$$

$$G_{y_2 u_1} = G_{y_2 u_2} G_{y_1 u_1} = \frac{n_1 n_2}{d_1 d_2}$$