CDS 101/110a: Lecture 3.1
Linear Systems

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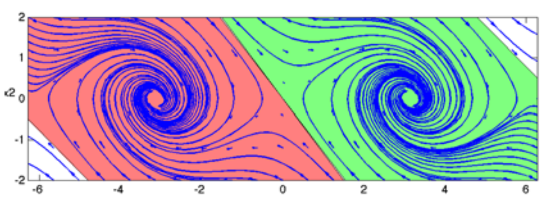
Goals:

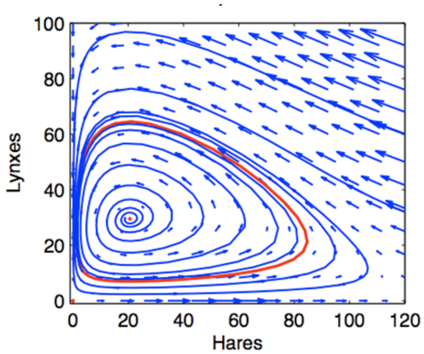
- Describe linear system models: properties, examples, and tools
  - Convolution equation describing solution in response to an input
  - Step response, impulse response
  - Frequency response

Reading:

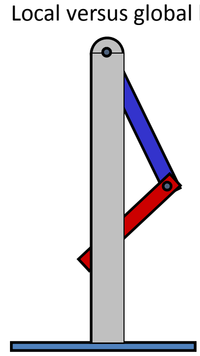
- Åström and Murray, Analysis and Design of Feedback Systems, Ch 5

Summary: Stability and Performance

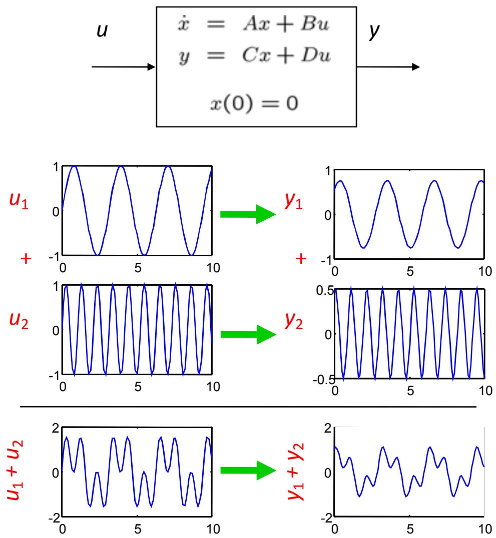




- Key topics
  - Stability of equilibrium points
  - Eigenvalues determine stability for linear systems
  - Local versus global behavior



Linear Systems

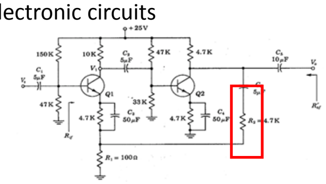


- Input/output linearity at x(0) = 0
  - Linear systems are linear in initial condition and input ⇒ need to use x(0) = 0 to add outputs together
  - For different initial conditions, you need to be more careful
- Linear system ⇒ step response and frequency response scale with input amplitude
  - 2X input ⇒ 2X output
  - Allows us to use ratios and percentages in step or frequency response. *These are independent of input amplitude*
  - Limitation: input saturation ⇒ only holds up to certain input amplitude

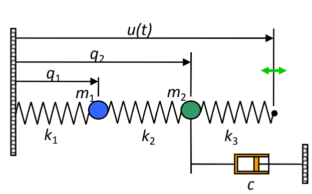
Why are Linear Systems Important?

Many important *examples*

- Electronic circuits



  - Especially true after feedback
  - Frequency response is key performance specification
- Many mechanical systems



Many important *tools*

- Frequency and step response,
  - Traditional tools of control theory
  - Developed in 1930's at Bell Labs
- Classical control design toolbox
  - Nyquist plots, gain/phase margin
  - Loop shaping

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- Optimal control and estimators
  - Linear quadratic regulators
  - Kalman estimators

CDS 110b

- Robust control design
  - H_∞ control design
  - μ analysis for structured uncertainty

CDS 212

Solutions of Linear Systems: The Matrix Exponential

$$\dot{x} = Ax + Bu \qquad y = Cx + Du \longrightarrow y(t) = ???$$

- Scalar linear system, with no input

$$\dot{x} = ax, \quad y = cx \qquad x(0) = x_0 \longrightarrow x(t) = e^{at}x_0 \longrightarrow y(t) = ce^{at}x_0$$

- Matrix version, with no input

$$\dot{x} = Ax, \quad y = Cx \qquad x(0) = x_0 \longrightarrow x(t) = e^{At}x_0 \longrightarrow \boxed{y(t) = Ce^{At}x_0}$$

sys=ss(A,B,C,D);
initial(sys,x0);

- Matrix exponential
  - Analog to the scalar case; defined by series expansion:

$$e^M = I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \cdots$$

P = expm(M)

Response to inputs: Discrete-time

- **First, with no input:** $x[k+1] = A_d x[k]$

$$x[1] = A_d x[0]$$
$$x[2] = A_d^2 x[0]$$
$$\vdots$$
$$x[k] = A_d^k x[0]$$

- **Now, with an input:**

$$x[k+1] = A_d x[k] + B_d u[k]$$

then:

$$x[1] = A_d x[0] + B_d u[0]$$
$$x[2] = A_d^2 x[0] + A_d B_d u[0] + B_d u[1]$$
$$x[3] = A_d^3 x[0] + A_d^2 B_d u[0] + A_d B_d u[1] + B_d u[2]$$
$$\vdots$$

- **Response due to input u[0] at time zero is**

$$\begin{bmatrix} x[1] \\ x[2] \\ x[3] \\ \vdots \end{bmatrix} = \begin{bmatrix} B_d \\ A_d B_d \\ A_d^2 B_d \\ \vdots \end{bmatrix}$$

- **Response at time *k* is the sum of response to all previous controls:**

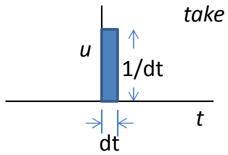
$$x[k+1] = A_d^k x[0] + B_d u[k] + A_d B_d u[k-1] + A_d^2 B_d u[k-2] \; \vdots$$

Back to continuous time…

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned} \longrightarrow y(t) = \underbrace{Ce^{At}x(0)}_{\text{homogeneous}} + ???$$

- What is the "impulse response" due to u(t)=δ(t)?

take limit as dt→0 but keep unit area



- Apply this unit impulse to the system (with x(0)=0):

$$x(0^+) = \int_{0^-}^{0^+} (Ax + Bu)\mathrm{d}t = B$$

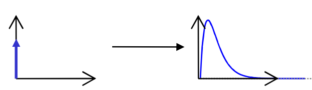
$$\Rightarrow \quad x(t) = e^{At}B$$

- Analogous to discrete-time response to input at time zero

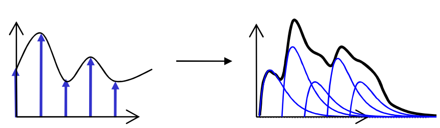
Response to inputs: Convolution

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned} \longrightarrow y(t) = \underbrace{Ce^{At}x(0)}_{\text{homogeneous}} + ???$$

- *Impulse response*, $h(t) = Ce^{At}B$
  - Response to input "impulse"
  - Equivalent to "Green's function"



- Linearity ⇒ compose response to arbitrary $u(t)$ using *convolution*
  - Decompose input into "sum" of shifted impulse functions
  - Compute impulse response for each
  - "Sum" impulse response to find $y(t)$
  - Take limit as dt→ 0



- Complete solution: use integral instead of "sum"

$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau + Du(t)$$

- linear with respect to initial condition *and* input
- 2X input ⇒ 2X output when $x(0) = 0$

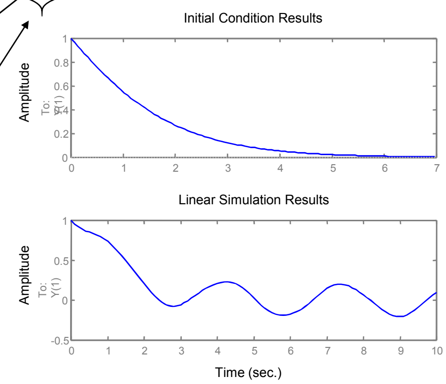
Matlab/Python Tools for Linear Systems

$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

```
A = [-1 1; 0 -1];  B = [0; 1];
C = [1 0]; D = [0];
x0 = [1; 0.5];

sys = ss(A,B,C,D);
initial(sys, x0);
impulse(sys);

t = 0:0.1:10;
u = 0.2*sin(5*t) + cos(2*t);
lsim(sys, u, t, x0);
```

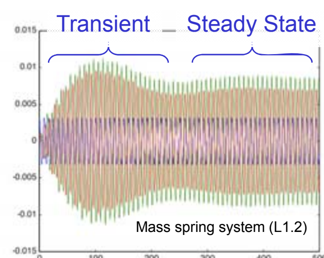


- Other MATLAB commands
  - gensig, square, sawtooth – produce signals of diff. types
  - step, impulse, initial, lsim – time domain analysis
  - bode, freqresp, evalfr – frequency domain analysis

ltiview – linear time invariant system plots

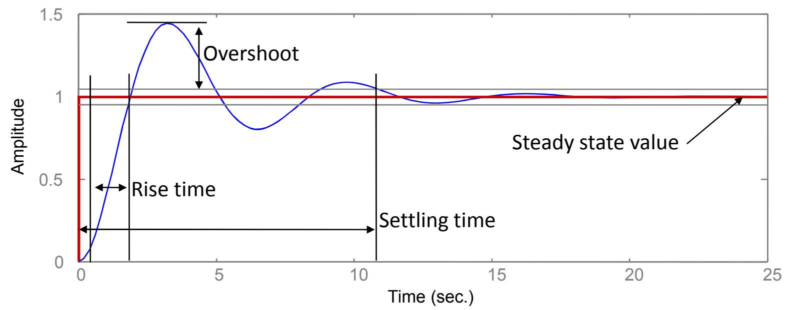
Input/Output Performance

- Return to system with inputs
  - How does system respond to changes in input values?
  - Transient response:
  - Steady state response:
- Characterize response in terms of
  - Impulse response
  - Step response
  - Frequency response
- Stability vs input/output performance
  - Systems that are close to instability typically exhibit poor input/output performance (slow convergence and/or "ringing" – a highly oscillatory response to [non-periodic] inputs)



Step Response

- Output characteristics in response to a "step" input
  - Rise time: time required to move from 5% to 95% of final value
  - Overshoot: ratio between amplitude of first peak and steady state value
  - Settling time: time required to remain w/in *p*% (usually 2%) of final value
  - Steady state value: final value at t = ∞



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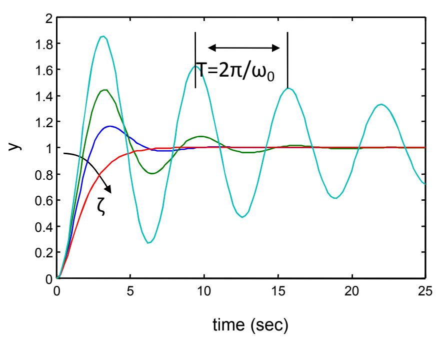
Second Order Systems

- If you understand response of first and second order systems, you understand the response for any order (eigenvalues of A are either real or complex)
  - Exception is non-diagonalizable A (non-trivial Jordan form)

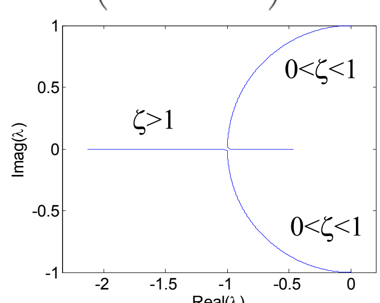
$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = u \quad \leftrightarrow \quad \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$



- Analytical formulas exist for overshoot, rise time, settling time, etc

For $\zeta < 1$, eigenvalues at

$$\left(-\zeta \pm j\sqrt{1-\zeta^2}\right)\omega_0$$



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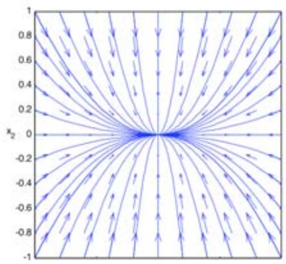
Eigenstructure of Linear Systems

Real e-values

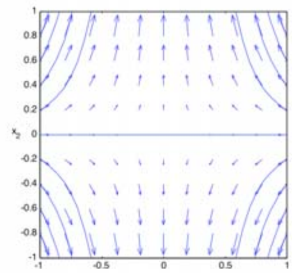
$Re(\lambda_i) < 0$



Real e-values

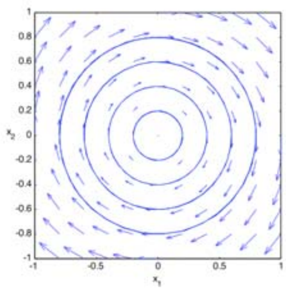
$Re(\lambda_i) < 0$

$Re(\lambda_i) > 0$



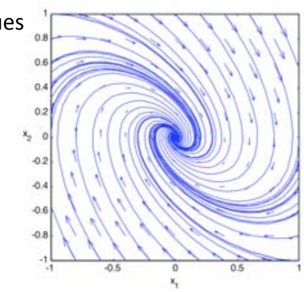
Complex e-values

$Re(\lambda_i) = 0$



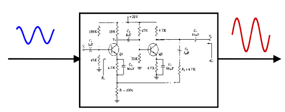
Complex e-values

$Re(\lambda_i) < 0$

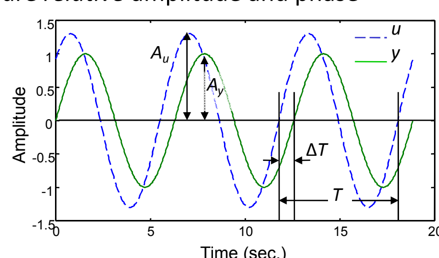


Frequency Response

- Measure the *steady state* response of the system to sinusoidal input
  - Example: audio amplifier – would like consistent ("flat") amplification between 20 Hz & 20,000 Hz
  - Individual sinusoids are good *test signals* for measuring performance in many systems (e.g., seasonal cycles in temperature)



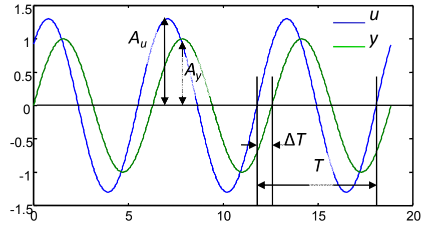
- Approach: plot input and output, measure *relative* amplitude and phase
  - Use MATLAB or SIMULINK to generate response of system to sinusoidal output
  - Gain = A_y/A_u
  - Phase = $2\pi \cdot \Delta T/T$



- May not work for *nonlinear* systems
  - System nonlinearities can cause *harmonics* to appear in the output
  - Amplitude and phase may not be well-defined
  - For *linear* systems, frequency response is always well defined

Computing Frequency Responses

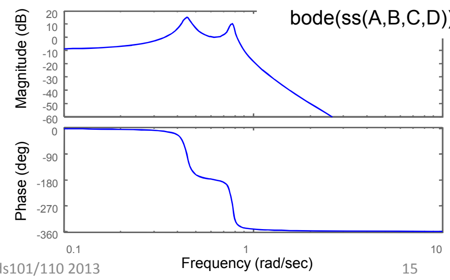
- Technique #1: plot input and output, measure relative amplitude and phase
  - Generate response of system to sinusoidal output
  - Gain = A_y/A_u
  - Phase = $2\pi \cdot \Delta T/T$
  - For *linear* system, gain and phase don't depend on the input amplitude



- Technique #2 (linear systems): use bode (or freqresp) command
  - Assumes linear dynamics in state space form:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

  - Gain plotted on log-log scale
    - dB = 20 $\log_{10}$ (gain)
  - Phase plotted on linear-log scale



Calculating Frequency Response from convolution equation… (more later)

- Convolution equation describes response to any input; use this to look at response to sinusoidal input: $u(t) = A\sin(\omega t) = \frac{A}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$

u(t)

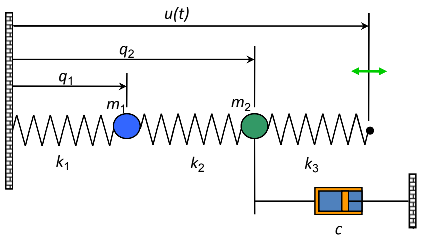
$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Be^{i\omega\tau}d\tau$$
$$= e^{At}x(0) + e^{At}\int_0^t e^{(i\omega I - A)\tau}Bd\tau$$
$$= e^{At}x(0) + e^{At}(i\omega I - A)^{-1}e^{(i\omega I-A)\tau}\Big|_{\tau=0}^{t}B$$
$$= e^{At}x(0) + e^{At}(i\omega I - A)^{-1}\left(e^{(i\omega I-A)t} - I\right)B$$
$$= \underbrace{e^{At}\left(x(0) - (i\omega I - A)^{-1}B\right)}_{\text{Transient (decays if stable)}} + \underbrace{(i\omega I - A)^{-1}B}_{\text{Ratio of response/input}}e^{i\omega t}$$

$$y(t) = Cx(t) + Du(t)$$
$$= Ce^{At}\left(x(0) - (i\omega I - A)^{-1}B\right) + \boxed{\left(C(i\omega I - A)^{-1}B + D\right)}e^{i\omega t}$$

"Frequency response"

Spring Mass System

Frequency response: $C(j\omega I-A)^{-1}B+D$



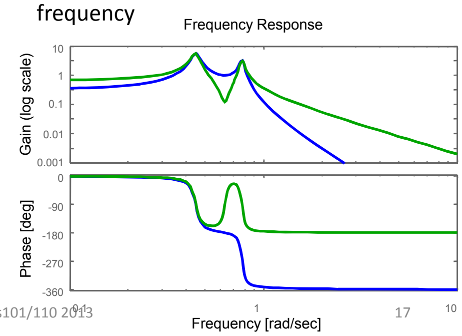
$$\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \\ \dot q_1 \\ \dot q_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m} & \frac{k_2}{m} & 0 & 0 \\ \frac{k_2}{m} & -\frac{k_2+k_3}{m} & 0 & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \dot q_1 \\ \dot q_2 \end{bmatrix}$$

With $k_1 = k_2 = 1,\ m = 1,\ c = 0$

$$v_{1,2} = \begin{bmatrix} 1 \\ 1 \\ \pm 1i \\ \pm 1i \end{bmatrix} \qquad v_{3,4} = \begin{bmatrix} 1 \\ -1 \\ \pm\sqrt{2}i \\ \mp\sqrt{2}i \end{bmatrix}$$

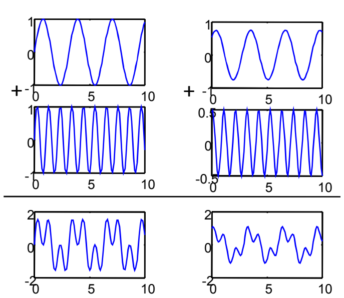
Eigenvalues of A:

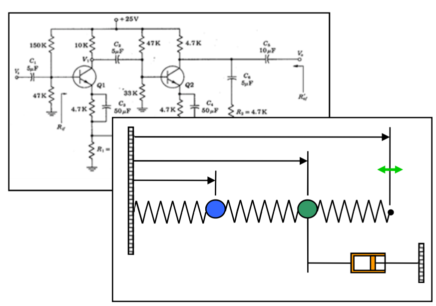
- For zero damping, $\pm j\omega_1$ and $\pm j\omega_2$
- ω_1 and ω_2 correspond to the two peaks in the frequency response
- The eigenvectors for these eigenvalues give the *mode shape*:
  - In-phase motion for the lower frequency
  - Out-of phase motion for the higher frequency



Summary: Linear Systems

u →
$$\dot x = Ax + Bu$$
$$y = Cx + Du$$
$$x(0) = 0$$
→ y





$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

- Properties of linear systems
  - Linearity with respect to initial condition and inputs
  - Stability characterized by eigenvalues
  - Many applications and tools available
  - Provide local description for nonlinear systems