CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 101

R. M. Murray — **Problem Set #4** — Issued: 22 Oct 2012
Fall 2012 — Due: 31 Oct 2012

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. Consider the normalized dynamics of an inverted pendulum described by the model

$$\frac{dx_1}{dt} = x_2, \qquad \frac{dx_2}{dt} = \sin x_1 + u \cos x_1,$$

where x_1 is the angular deviation from the upright position (θ), x_2 is the angular rate ($\dot{\theta}$) and u is the (scaled) acceleration of the pivot, as shown in Figure 4.16a.

(a) Show that the linearization of the dynamics around the (upward-pointing) equilibrium point $x_e = (0, 0)$ is given by

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u = Ax + Bu.$$

(b) Compute the eigenvalues of the linearized dynamics and show that this equilibrium point is unstable.

(c) Compute the reachability matrix for the system and show that the linearized system is reachable.

(d) Determine a state feedback control law of the form $u = -Kx$ that gives a closed loop system with with the characteristic polynomial $s^2 + 2\zeta_0\omega_0 s + \omega_0^2$.

(e) Set $\omega_0 = 1$ and $\zeta_0 = 0.5$. Compute the eigenvalues for the resulting closed loop system and verify that the equilibrium point is no longer unstable.

(f) Simulate the response of the original nonlinear system from a set of initial conditions that each correspond to the system starting at rest ($\dot{\theta} = 0$) with initial angle $\theta(0)$ equal to 0.1 rad, 0.5 rad, 1 rad and 2 rad.

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CDS 110a

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1. Consider the normalized, linearized inverted pendulum which is described by

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u = Ax + Bu, \qquad y = \begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Cx$$

   Determine a state feedback and reference gain $u = Kx + k_r r$ that gives a closed loop system with unit static gain (steady-state output $y = r$) and with the characteristic polynomial $s^2 + 2\zeta_0\omega_0 s + \omega_0^2$.

2. Åström and Murray, Exercise 6.3.

3. Åström and Murray, Exercise 6.9. (Show explicitly; don't just cite Theorem 6.1.)

4. Åström and Murray, Exercise 6.10

   Assume that the A matrix is diagonalizable (the theorem is valid but hard to prove with a non-trivial Jordan form).

5. Åström and Murray, Exercise 6.12

   Download the file `bike_linmod.m` from the course web site, which contains the parameters for the bicycle and generates the matrices M, C, K_0 and K_2 in equation (3.7) of the text.

   Find the controller gains corresponding to choosing the final pair of complex poles at $-1 \pm i$ as stated in the text, and also with these poles at $-2 \pm 2i$ and $-5 \pm 5i$. In addition to calculating the state feedback gains, solve for the reference gain k_r as well! When simulating the response to a step change in the steering reference of 0.002 rad, plot both the steering angle output δ and the torque command.