CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 101

D. MacMartin and R. Murray
Fall 2012

Problem Set #1

Issued: 1 Oct 12
Due: 10 Oct 12

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

Reminder: Late homework will not be accepted without a note from the dean or the health center.

1. Åström and Murray, Exercise 1.2

2. Consider the cruise-control example discussed in class, with

$$m\dot{v} = -av + u + w$$

where u is the control input (force applied by engine) and w the disturbance input (force applied by hill, etc.), which will be ignored below ($w = 0$). An *open-loop* control strategy to achieve a given reference speed v_{ref} would be to choose

$$u = \hat{a}v_{\text{ref}}$$

where $\hat{a}$ is your estimate of a, which may not be accurate.

(a) Compute the steady-state response for both the open-loop strategy above, and for the feedback law

$$u = -k_p(v - v_{\text{ref}})$$

and compare the steady-state (with $w = 0$) as a function of $\beta = a/\hat{a}$ when $k_p = 10\hat{a}$. (You should solve the problem analytically, and then plot the response $v_{\text{ss}}/v_{\text{ref}}$ as a function of β.)

(b) Now consider a proportional-integral control law

$$u = -k_p(v - v_{\text{ref}}) - k_i \int_0^t (v - v_{\text{ref}})dt$$

and again compute the steady state solution (assuming stability) and compare the response with the proportional gain case from above. (Note that if you define $q = \int_0^t (v - v_{\text{ref}})dt$ then $\dot{q} = v - v_{\text{ref}}$.)

3. Åström and Murray, Exercise 2.6, parts (a) and (b)

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1. Åström and Murray, Exercise 1.2
2. Åström and Murray, Exercise 2.1
3. Consider the cruise-control example discussed in class, with

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(a) Compute the steady-state response for both the open-loop strategy above, and for the feedback law

$$u = -k_p(v - v_{\text{ref}})$$

and compare the steady-state (with $w = 0$) as a function of $\beta = a/\hat{a}$ when $k_p = 10\hat{a}$. (You should solve the problem analytically, and then plot the response $v_{\text{ss}}/v_{\text{ref}}$ as a function of β.)

(b) Now consider a proportional-integral (PI) control law

$$u = -k_p(v - v_{\text{ref}}) - k_i \int_0^t (v - v_{\text{ref}})dt$$

and again compute the steady state solution (assuming stability) and compare the response with the proportional gain case from above. (Note that if you define $q = \int_0^t (v - v_{\text{ref}})dt$ then $\dot{q} = v - v_{\text{ref}}$.)

(c) Next, simulate the response of the system (using `ode45` in Matlab or `odeint` in SciPy or something similar) with the PI control law above with $m = 1$, $a = 0.1$, $w = 0$, and "input" to the system of $v_{\text{ref}} = \sin(\omega t)$, for ω=0.01, 0.1, 1, and 10 rad/sec. In each case, you should simulate at least 10 cycles; after some initial transient, the response should be periodic. Compute the peak-to-peak amplitude of the final period for the error $v - v_{\text{ref}}$, and plot this as a function of frequency on a log-log scale, for the following control gains:

   i. $k_p = 1,\ k_i = 0$
   ii. $k_p = 1,\ k_i = 1$
   iii. $k_p = 1,\ k_i = 10$

   (If you want to see interesting behaviour, simulate the final case at $\omega = 3.3\,\text{rad/sec}$ as well.)

4. Consider a damped spring–mass system with dynamics

$$m\ddot{q} + c\dot{q} + kq = F.$$

Let $\omega_0 = \sqrt{k/m}$ be the natural frequency and $\zeta = c/(2\sqrt{km})$ be the damping ratio.

(a) Show that by rescaling the equations, we can write the dynamics in the form

$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = \omega_0^2 u, \tag{S1.1}$$

where $u = F/k$. This form of the dynamics is that of a linear oscillator with natural frequency ω_0 and damping ratio ζ.

(b) Show that the system can be further normalized and written in the form

$$\frac{dz_1}{d\tau} = z_2, \qquad \frac{dz_2}{d\tau} = -z_1 - 2\zeta z_2 + v. \tag{S1.2}$$

The essential dynamics of the system are governed by a single damping parameter ζ. The *Q-value* defined as $Q = 1/2\zeta$ is sometimes used instead of ζ.

(c) Show that the solution for the unforced system ($v = 0$) with no damping ($\zeta = 0$) is given by

$$z_1(\tau) = z_1(0)\cos\tau + z_2(0)\sin\tau, \qquad z_2(\tau) = -z_1(0)\sin\tau + z_2(0)\cos\tau.$$

Invert the scaling relations to find the form of the solution $q(t)$ in terms of $q(0)$, $\dot{q}(0)$ and ω_0.

(d) Consider the case where $\zeta = 0$ and $u(t) = \sin\omega t$, $\omega > \omega_0$. Solve for $z_1(\tau)$, the normalized output of the oscillator, with initial conditions $z_1(0) = z_2(0) = 0$ and use this result to find the solution for $q(t)$.

(Parts (a) and (b) are from AM 2.6.)