CDS 101/110a: Lecture 9-1
PID Control

Douglas G. MacMartin

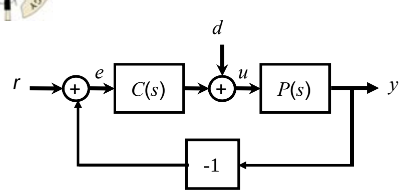
Goals:

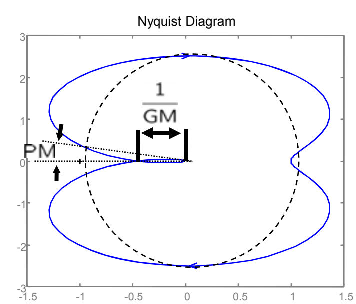
- Introduction to frequency-domain performance specification
- Show how to use PID (Proportional + Integral + Derivative) feedback to achieve a performance specification

Reading:

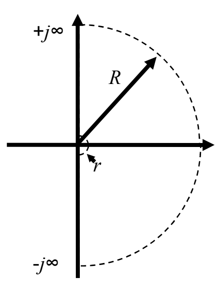
- Åström and Murray, *Feedback Systems,* Ch 10
- *Advanced*: Lewis, Chapters 12-13

Last Week: Loop Analysis





- Nyquist criteria for loop stability
- Gain, phase margin for robustness



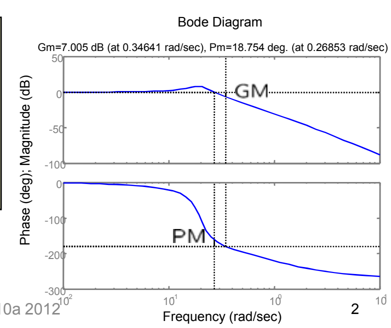
Thm (Nyquist).

P # RHP poles of $L(s)$

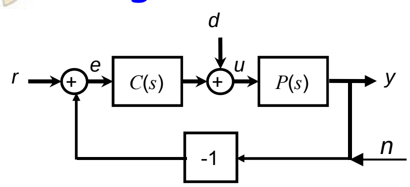
N # CW encirclements

Z # RHP zeros of $1+L$

$$Z = N + P$$



Design based on loop transfer function



$$L(s) = P(s)C(s)$$

$$H_{er} = \frac{1}{1+L} \qquad H_{yr} = \frac{L}{1+L}$$

$$H_{yd} = \frac{P}{1+L} \qquad H_{yn} = \frac{-L}{1+L}$$

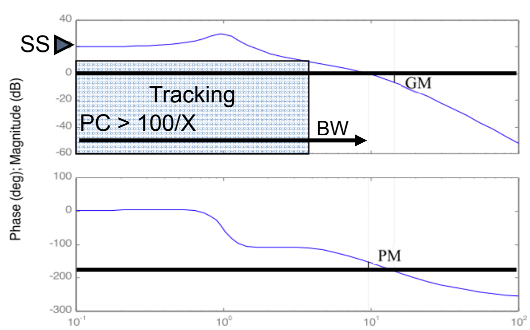
- Stability depends only on *L = PC* (last week)
  - Robustness requires reasonable gain and phase margin
- Performance depends (mostly) on *L = PC*
  - When L is large, tracking performance and disturbance rejection is good
  - When L is small, sensor noise rejection is good, actuator response is small.
  - Typically care about the tracking and disturbance response at low frequencies
  - If gain or phase margin is small, tend to get large overshoot and ringing

Frequency Domain Performance Specifications

- Specify bounds on the loop transfer function to guarantee desired performance
  - Steady state error: $H_{er}(0) = \frac{1}{(1+L(0))} \simeq 1/L(0)$

    ⇒ sets zero frequency ("DC") gain ▶
  - Tracking: Error less than X% up to frequency ω

    ⇒ Determines gain bound |1+L|>100/X



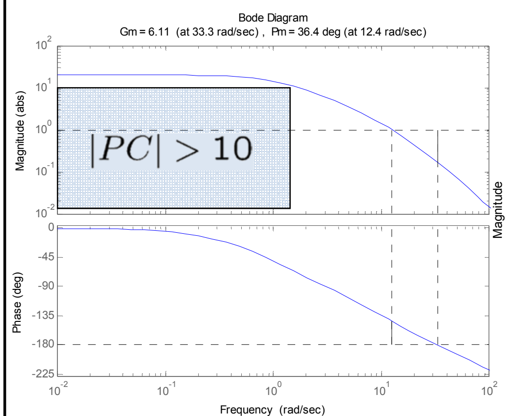
- Bandwidth:
  - assuming ~90˚ phase margin

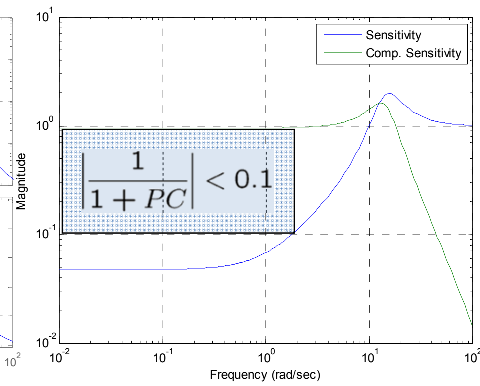
$$\frac{L}{1+L}(i\omega_c) = \left|\frac{1}{1+i}\right| = \frac{1}{\sqrt{2}}$$

⇒ sets loop crossover frequency

Example:

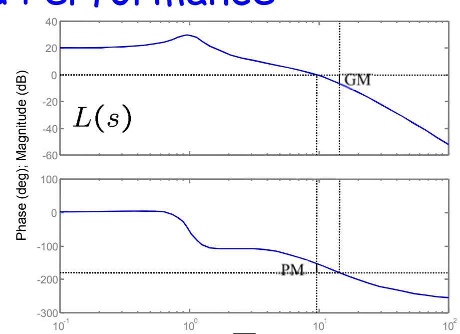
- Suppose $L(s) = \dfrac{20}{(s+1)(s/10+1)(s/100+1)}$

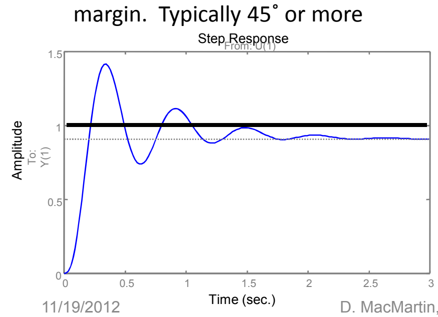


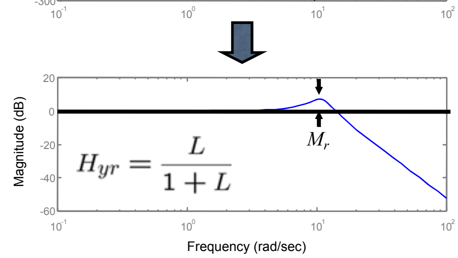


Relative Stability: Robustness and Performance

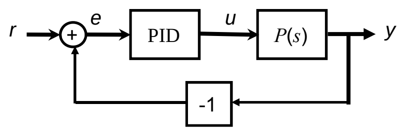
- Loop transfer function close to -1 gives poor robustness *and* performance:
  - System can be stable but still have bad response at certain frequencies
  - Typically occurs if system has low phase margin $\Rightarrow$ get resonant peak in closed loop $\Rightarrow$ overshoot; poor step response
  - Solution: specify minimum phase margin. Typically 45˚ or more







Overview: PID control

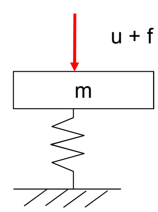


$$u = k_p e + k_i \int e\,\mathrm{d}t + k_d \dot{e}$$

$$C(s) = \frac{k_i}{s} + k_p + k_d s$$

- Intuition
  - Proportional term: provides inputs that correct for “current” errors
  - Integral term: ensures steady state error goes to zero
  - Derivative term: provides “anticipation” of upcoming changes
- A bit of history on “three term control”
  - First appeared in 1922 paper by Minorsky: “Directional stability of automatically steered bodies” under the name “three term control”
  - Also realized that “small deviations” (linearization) could be used to understand the (nonlinear) system dynamics under control
- Utility of PID
  - PID control is most common feedback structure in engineering systems
  - For many systems, only need PI or PD (special case)
  - Many tools for tuning PID loops and designing gains

Time-domain motivation



- Mass-spring system: $m\ddot{z} + c\dot{z} + kz = u + f$
- PD control: $u = -(k_p z + k_d \dot{z})$
- Closed-loop: $m\ddot{z} + (c + k_d)\dot{z} + (k + k_p)z = f$
  - Derivative gain acts like increasing damping
    - Increases system stability (greater phase margin)
  - Proportional gain acts like increasing stiffness
    - No matter how large the stiffness, still a non-zero response to disturbance force
  - Steady-state (for constant disturbance force *f*)

$$\dot{z} = 0 \quad \Rightarrow \quad \lim_{t\to\infty} z(t) = \frac{1}{k + k_p} f, \text{ and } \lim_{t\to\infty} u(t) = -\frac{k_p}{k + k_p} f$$

- Integral control: $\dot{q} = z$

$$u = -(k_i q + k_p z + k_d \dot{z})$$

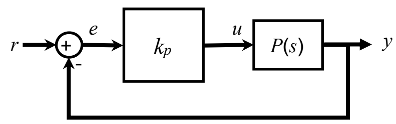
  - Steady-state (assuming stability) then for constant *f*

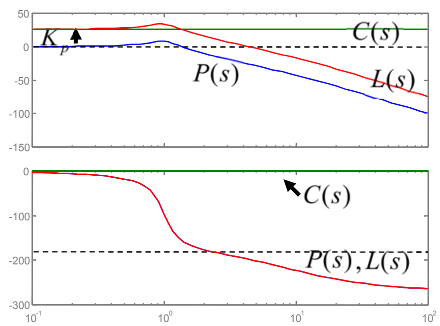
$$\dot{q} = 0 \quad \Rightarrow \quad \lim_{t\to\infty} z = 0, \text{ and } \lim_{t\to\infty} u(t) = -f$$

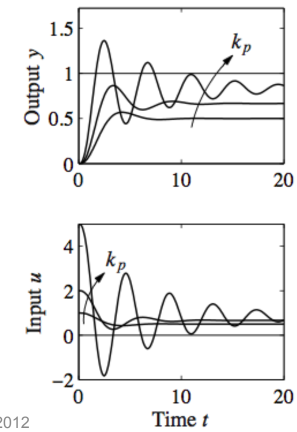
Proportional Feedback

$k_p > 0$ if $P(0) > 0$

- Simplest controller choice: $u = k_p e$
  - Effect: lifts gain with no change in phase
  - Good for plants with low phase up to desired bandwidth
  - Bode: shift gain up by factor of k_p
  - Step response: better steady state error, but with decreasing stability

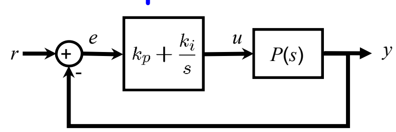




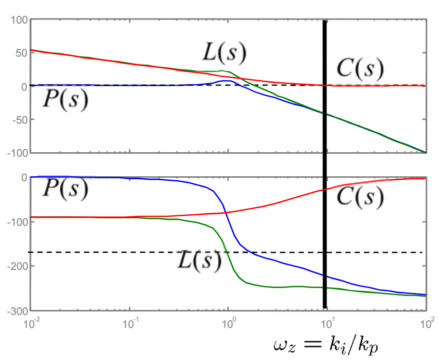


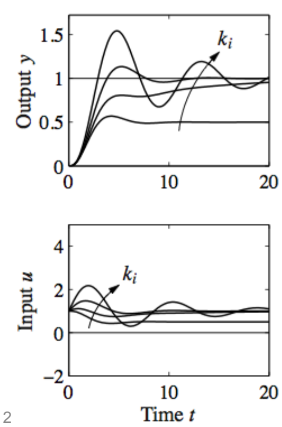
Proportional + Integral Compensation

- Use to eliminate steady state error
  - Effect: lifts gain at low frequency
  - Gives *zero* steady state error
  - Bode: infinite SS gain + phase lag
  - Step response: zero steady state error, with smaller settling time, but more overshoot

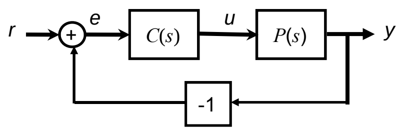


$k_p > 0$, $k_i > 0$

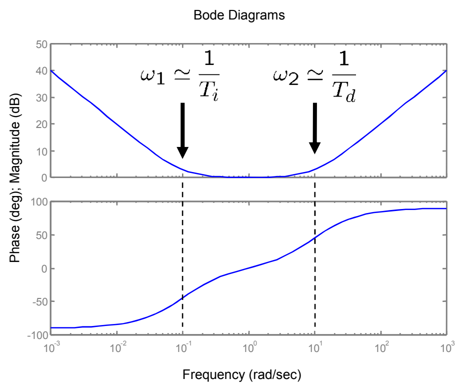


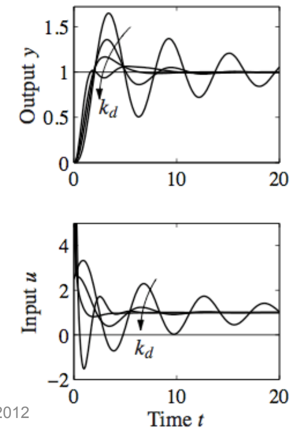


Proportional + Integral + Derivative (PID)

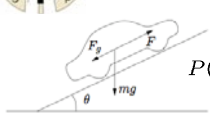


$$C(s) = k_p + k_i \frac{1}{s} + k_d s$$
$$= k\left(1 + \frac{1}{T_i s} + T_d s\right)$$
$$= (kT_d)\frac{(s+\alpha_i)(s+\alpha_d)}{s}$$



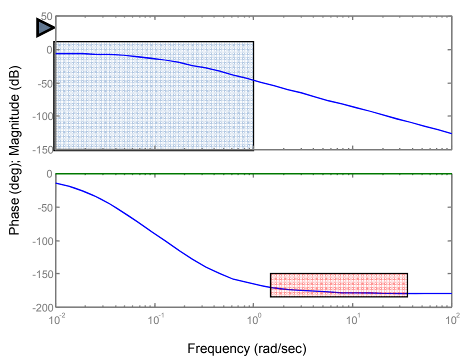


Example: Cruise Control using PID - Specification

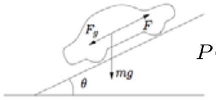


$$P(s) = \frac{1/m}{s + b/m} \times \frac{r}{s+a}$$

- Performance Specification
  - $\leq$ 1% steady state error
    - $\Rightarrow$ Zero frequency gain > 100
  - $\leq$ 10% tracking error up to 1 rad/sec
    - $\Rightarrow$ Gain > 10 from 0-1 rad/sec
  - $\geq$ 45˚ phase margin
    - $\Rightarrow$ Gives good relative stability
    - $\Rightarrow$ Provides robustness to uncertainty
    - $\Rightarrow$ But overshoot will be ~25%
- Observations
  - Purely proportional gain won't work: to get gain above desired level will not leave adequate phase margin
  - Need to increase the phase from ~0.5 to 2 rad/sec and increase gain as well



Example: Cruise Control using PID - Design



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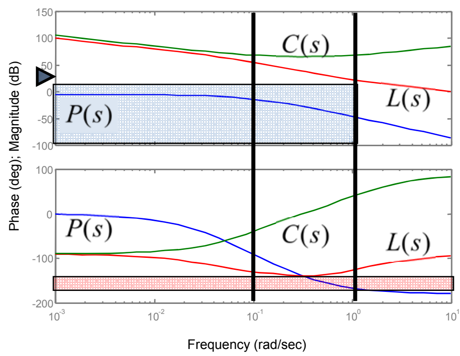
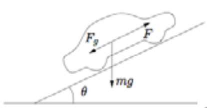


- Approach
  - Use proportional gain to give desired tracking performance
  - Use integral gain to make steady state error small (zero, in fact)
  - Use derivative action to increase phase lead in the cross over region
- Controller
  - Ti = 1/0.1; Td = 1/1; k = 2000

$$C(s) = 2000\frac{s^2 + 1.1s + 0.1}{s}$$
$$= 2200 + \frac{200}{s} + 2000s$$

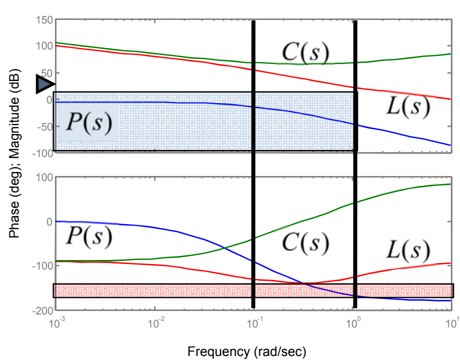
- Closed loop system
  - Very high steady state gain
  - Adequate tracking @ 1 rad/sec
  - ~80° phase margin
  - Verify with Nyquist + Gang of 4

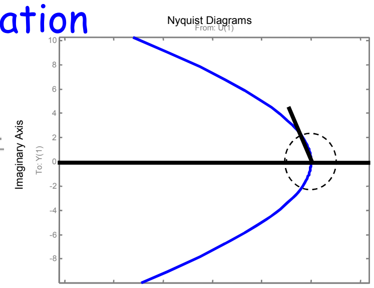
Example: Cruise Control using PID - Verification

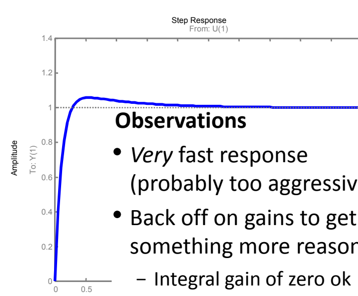


$$P(s) = \frac{1/m}{s + b/m} \times \frac{r}{s+a}$$

$$C(s) = 2000\frac{s^2 + 1.1s + 0.1}{s}$$





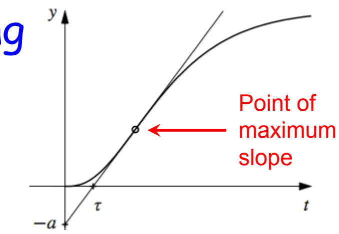


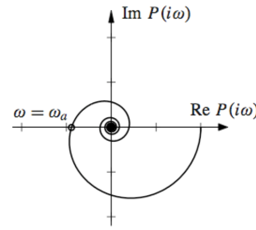
Observations

- *Very* fast response (probably too aggressive)
- Back off on gains to get something more reasonable
  - Integral gain of zero ok
  - Lower rate fdbk gain ok

Automated PID Tuning

- Zeigler-Nichols step response method
  - Design PID gains based on step response
  - Measure maximum slope + intercept
  - Works OK for many plants (but underdamped)
  - Good way to get a first cut controller
- Ziegler-Nichols frequency response method
  - Increase gain until system goes unstable
  - Use critical gain and frequency as parameters
- Variations
  - Modified formulas (see text) give better response
  - Relay feedback: provides automated way to obtain critical gain, frequency





Type	k_p	T_i	T_d
P	1/a		
PI	0.9/a	3τ	
PID	1.2/a	2τ	0.5τ

(a) Step response method

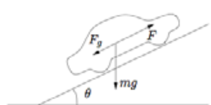
Type	k_p	T_i	T_d
P	$0.5k_c$		
PI	$0.4k_c$	$0.8T_c$	
PID	$0.6k_c$	$0.5T_c$	$0.125T_c$

(b) Frequency response method

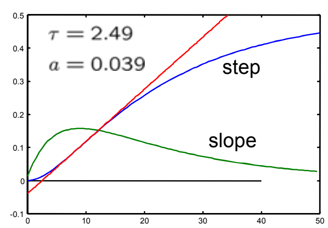
$$k_p = \frac{0.15\tau + 0.35T}{K\tau} \quad \left(\frac{0.9T}{K\tau}\right), \quad k_i = \frac{0.46\tau + 0.02T}{K\tau^2} \quad \left(\frac{0.3T}{K\tau^2}\right),$$

$$k_p = 0.22k_c - \frac{0.07}{K} \quad \left(0.4k_c\right), \quad k_i = \frac{0.16k_c}{T_c} + \frac{0.62}{KT_c} \quad \left(\frac{0.5k_c}{T_c}\right).$$

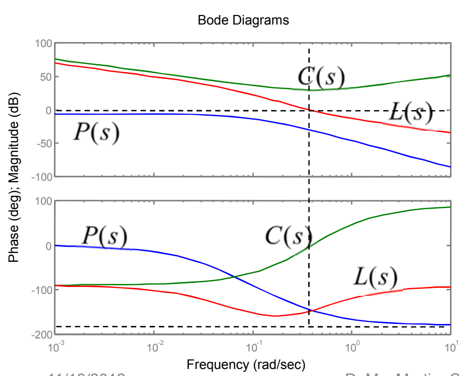
Example: PID cruise control

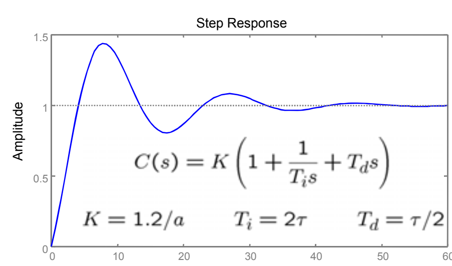


$$P(s) = \frac{1/m}{s + b/m} \times \frac{r}{s+a}$$



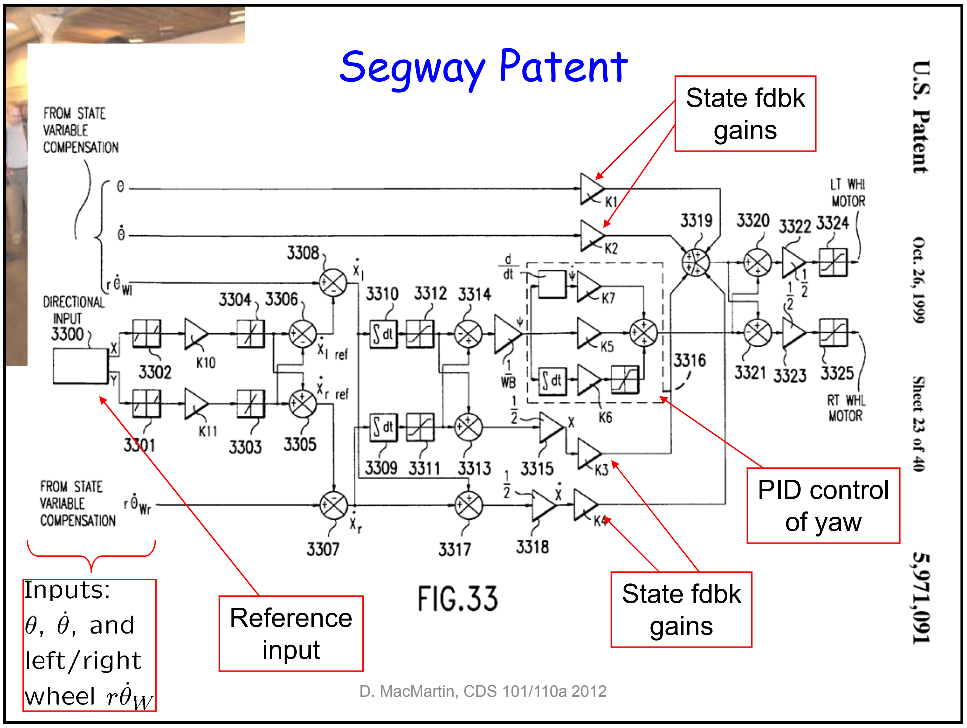
- Ziegler-Nichols design for cruise controller
  - Plot step response, extract τ and a, compute gains





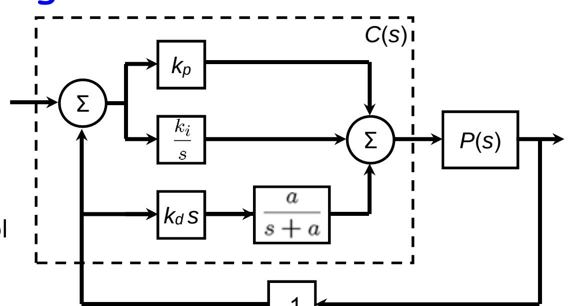
- Result: *sluggish* → increase loop gain + more phase margin (shift zero)

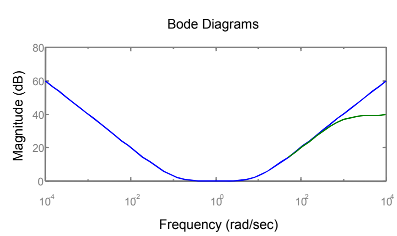
Segway Patent



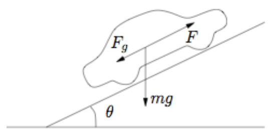
Implementing Derivative Action

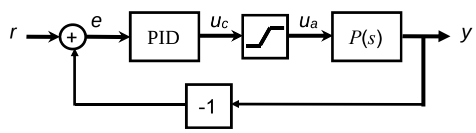
- Problems with derivatives
  - High frequency noise amplified by derivative term
  - Step inputs in reference can cause large inputs
  - Shows up in Gang of Four...
- Solution: modified PID control
  - Use high frequency rolloff in derivative term
    - first order filter will give finite gain at high frequency
    - use higher order filter if needed
  - Don't feed reference signal through derivative block
    - Useful when reference has unwanted high frequency content
    - Alternative solution: reference shaping via two DOF design ($F(s)$ block)
  - Many other variations (see AM08 + refs)



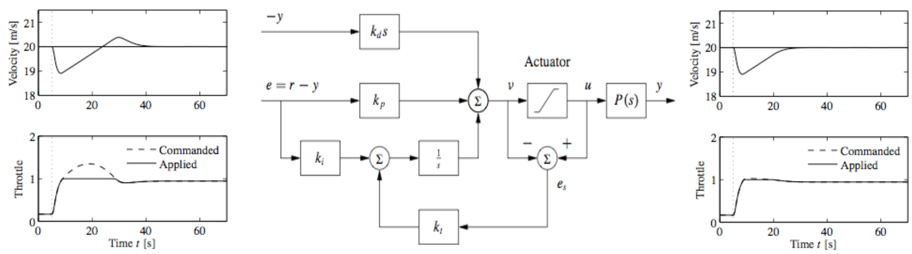


Windup and Anti-Windup Compensation





- Problem
  - Limited magnitude input (saturation)
  - Integrator "winds up" ⇒ overshoot
- Solution
  - Compare commanded input to actual
  - Subtract off difference from integrator



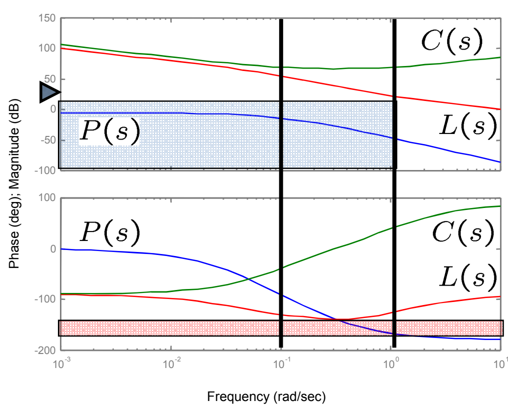
(a) Windup

(b) Anti-windup

Summary: Frequency Domain Design using PID

- Loop Shaping for Stability & Performance
  - Steady state error, bandwidth, tracking

$$H_{ue}(s) = k_p + k_i \frac{1}{s} + k_d s$$



- Main ideas
  - Performance specs give bounds on loop transfer function
  - Use controller to shape response
  - Gain/phase relationships constrain design approach
  - Standard compensators: proportional, PI, PID