



CDS 101/110a: Lecture 3.1 Linear Systems

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Goals:

- Describe linear system models: properties, examples, and tools
 - Convolution equation describing solution in response to an input
 - Step response, impulse response
 - Frequency response

Reading:

- Åström and Murray, Analysis and Design of Feedback Systems, Ch 5

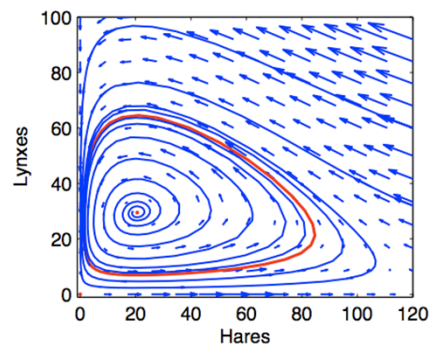
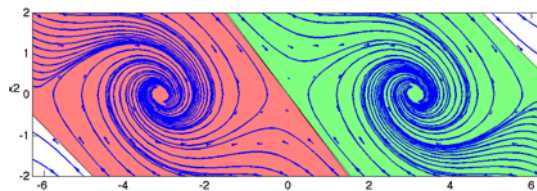
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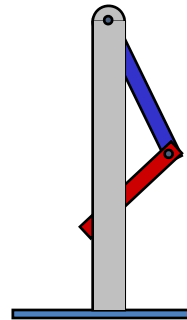


Summary: Stability and Performance



• Key topics


- Stability of equilibrium points
- Eigenvalues determine stability for linear systems
- Local versus global behavior



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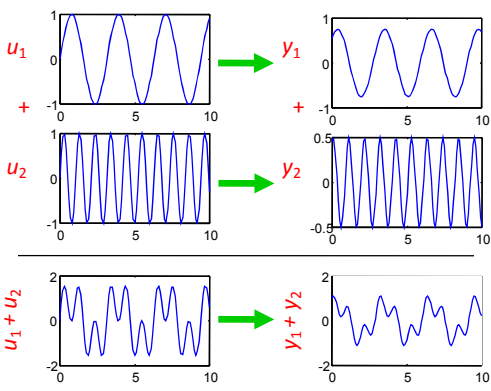
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Linear Systems

$$\begin{array}{c}
 u \rightarrow \left[\begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx + Du \\ x(0) = 0 \end{array} \right. \rightarrow y
 \end{array}$$



- Input/output linearity at $x(0) = 0$
 - Linear systems are linear in initial condition and input \Rightarrow need to use $x(0) = 0$ to add outputs together
 - For different initial conditions, you need to be more careful
- Linear system \Rightarrow step response and frequency response scale with input amplitude
 - 2X input \Rightarrow 2X output
 - Allows us to use ratios and percentages in step or frequency response. *These are independent of input amplitude*
 - Limitation: input saturation \Rightarrow only holds up to certain input amplitude

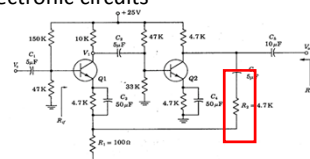
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Why are Linear Systems Important?

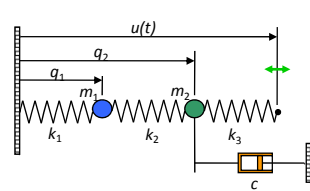
Many important examples

- Electronic circuits



- Especially true after **feedback**
- Frequency response is key performance specification


- Many mechanical systems



Many important tools

- Frequency and step response,
 - Traditional tools of control theory
 - Developed in 1930's at Bell Labs
- Classical control design toolbox
 - Nyquist plots, gain/phase margin
 - Loop shaping
- Optimal control and estimators
 - Linear quadratic regulators
 - Kalman estimators
- Robust control design
 - H_∞ control design
 - μ analysis for structured uncertainty

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Linearization Around an Equilibrium Point

$$\begin{aligned} \dot{x} &= f(x, u) & \dot{z} &= Az + Bu \\ y &= h(x, u) & y &= Cz + Du \end{aligned}$$

"Linearize" around $x=x_e$

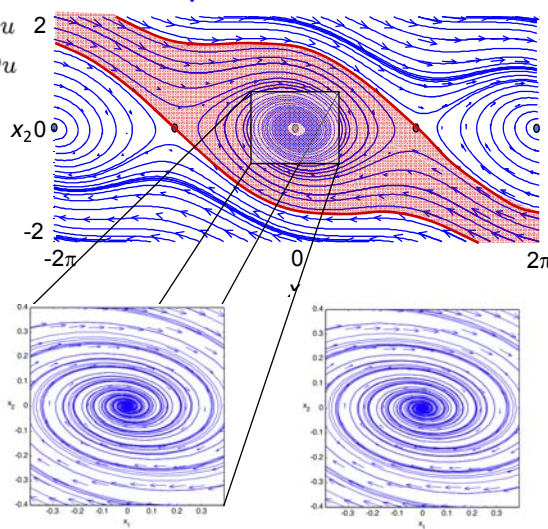
$$f(x_e, u_e) = 0 \quad y_e = h(x_e, u_e)$$

$$z = x - x_e \quad v = u - u_e \quad w = y - y_e$$

$$A = \left. \frac{\partial f}{\partial x} \right|_{(x_e, u_e)} \quad B = \left. \frac{\partial f}{\partial u} \right|_{(x_e, u_e)}$$


$$C = \left. \frac{\partial h}{\partial x} \right|_{(x_e, u_e)} \quad D = \left. \frac{\partial h}{\partial u} \right|_{(x_e, u_e)}$$

- Remarks
- In examples, this is often equivalent to small angle approximations, etc
- Only works *near* equilibrium point

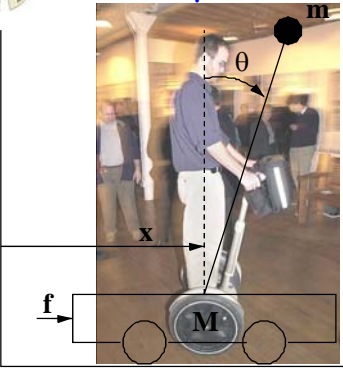


Full nonlinear model
Linear model (honest!)

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Example: Inverted Pendulum on a Cart



$$(M + m)\ddot{x} + ml \cos \theta \ddot{\theta} = -b\dot{x} + ml \sin \theta \dot{\theta}^2 + f$$


$$(J + ml^2)\ddot{\theta} + ml \cos \theta \dot{x} = -mgl \sin \theta$$

- State: $x, \theta, \dot{x}, \dot{\theta}$
- Input: $u = F$
- Output: $y = x$
- Linearize according to previous formula around $\theta = 0$

$$\frac{d}{dt} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2 g l^2}{J(M+m) + Mml^2} & \frac{-(J + ml^2)b}{J(M+m) + Mml^2} & 0 \\ 0 & \frac{mgl(M+m)}{J(M+m) + Mml^2} & \frac{-mlb}{J(M+m) + Mml^2} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \frac{J + ml^2}{J(M+m) + Mml^2} \\ \frac{ml}{J(M+m) + Mml^2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$$

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Solutions of Linear Systems: The Matrix Exponential

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \longrightarrow y(t) = ???$$

- Scalar linear system, with no input

$$\begin{aligned} \dot{x} &= ax \\ y &= cx \end{aligned} \quad x(0) = x_0 \longrightarrow x(t) = e^{at}x_0 \longrightarrow y(t) = ce^{at}x_0$$
- Matrix version, with no input


$$\begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned} \quad x(0) = x_0 \longrightarrow x(t) = e^{At}x_0 \longrightarrow y(t) = Ce^{At}x_0$$
- Matrix exponential
 - Analog to the scalar case; defined by series expansion:

$$e^M = I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \dots$$

`sys=ss(A,B,C,D);`
`initial(sys,x0);`

P = expm(M)

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Response to inputs: Discrete-time

- **First, with no input:** $x[k + 1] = A_d x[k]$

$$\begin{aligned} x[1] &= A_d x[0] \\ x[2] &= A_d^2 x[0] \\ &\vdots \\ x[k] &= A_d^k x[0] \end{aligned}$$
- **Now, with an input:** $x[k + 1] = A_d x[k] + B_d u[k]$


then:

$$\begin{aligned} x[1] &= A_d x[0] + B_d u[0] \\ x[2] &= A_d^2 x[0] + A_d B_d u[0] + B_d u[1] \\ x[3] &= A_d^3 x[0] + A_d^2 B_d u[0] + A_d B_d u[1] + B_d u[2] \\ &\vdots \end{aligned}$$
- **Response due to input u[0] at time zero is**

$$\begin{bmatrix} x[1] \\ x[2] \\ x[3] \\ \vdots \end{bmatrix} = \begin{bmatrix} B_d \\ A_d B_d \\ A_d^2 B_d \\ \vdots \end{bmatrix}$$
- **Response at time k is the sum of response to all previous controls:**

$$\begin{aligned} x[k + 1] &= A_d^k x[0] \\ &\quad + B_d u[k] \\ &\quad + A_d B_d u[k - 1] \\ &\quad + A_d^2 B_d u[k - 2] \\ &\quad \vdots \end{aligned}$$

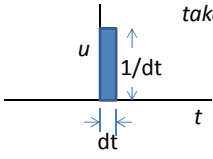
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Back to continuous time...

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad \longrightarrow \quad y(t) = \underbrace{Ce^{At}x(0)}_{\text{homogeneous}} + ???$$

- What is the "impulse response" due to $u(t)=\delta(t)$?
take limit as $dt \rightarrow 0$ but keep unit area




- Apply this unit impulse to the system (with $x(0)=0$):

$$x(0^+) = \int_{0^-}^{0^+} (Ax + Bu)dt = B$$

$$\Rightarrow x(t) = e^{At}B$$

- Analogous to discrete-time response to input at time zero

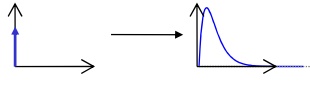
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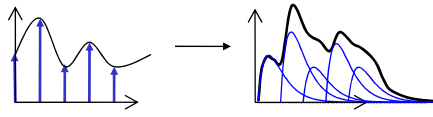
Response to inputs: Convolution

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad \longrightarrow \quad y(t) = \underbrace{Ce^{At}x(0)}_{\text{homogeneous}} + ???$$

- Impulse response, $h(t) = Ce^{At}B$**
 - Response to input "impulse"
 - Equivalent to "Green's function"




- Linearity \Rightarrow compose response to arbitrary $u(t)$ using convolution**
 - Decompose input into "sum" of shifted impulse functions
 - Compute impulse response for each
 - "Sum" impulse response to find $y(t)$
 - Take limit as $dt \rightarrow 0$



- Complete solution: use integral instead of "sum"**
 - linear with respect to initial condition *and* input
 - 2X input \Rightarrow 2X output when $x(0) = 0$

$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

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Matlab Tools for Linear Systems

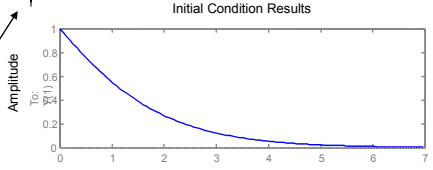
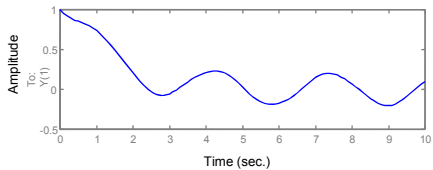
$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

```

A = [-1 1; 0 -1]; B = [0; 1];
C = [1 0]; D = [0];
x0 = [1; 0.5];


sys = ss(A,B,C,D);
initial(sys, x0);
impulse(sys);

t = 0:0.1:10;
u = 0.2*sin(5*t) + cos(2*t);
lsim(sys, u, t, x0);
                    
```


- Other MATLAB commands
 - gensig, square, sawtooth – produce signals of diff. types
 - step, impulse, initial, lsim – time domain analysis
 - bode, freqresp, evalfr – frequency domain analysis

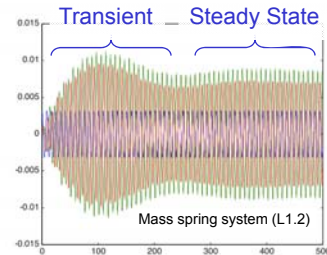
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Input/Output Performance


- Return to system with inputs
 - How does system respond to changes in input values?
 - Transient response:
 - Steady state response:
- Characterize response in terms of
 - Impulse response
 - Step response
 - Frequency response
- Stability vs input/output performance
 - Systems that are close to instability typically exhibit poor input/output performance (slow convergence and/or “ringing” – a highly oscillatory response to [non-periodic] inputs)






Mass spring system (L1.2)

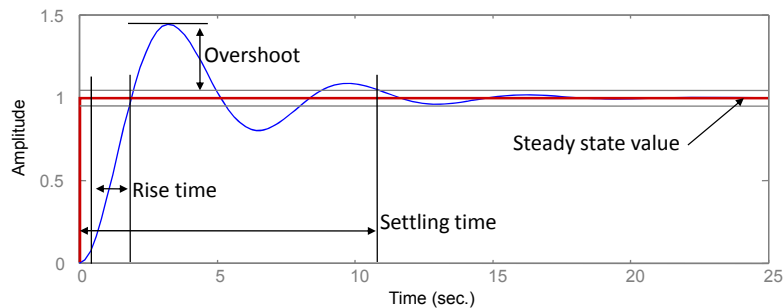
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Step Response

- Output characteristics in response to a “step” input
 - Rise time: time required to move from 5% to 95% of final value
 - Overshoot: ratio between amplitude of first peak and steady state value
 - Settling time: time required to remain w/in $p\%$ (usually 2%) of final value
 - Steady state value: final value at $t = \infty$



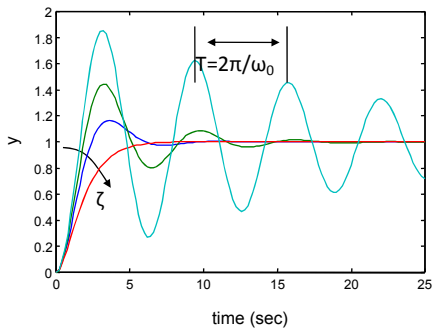


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Second Order Systems

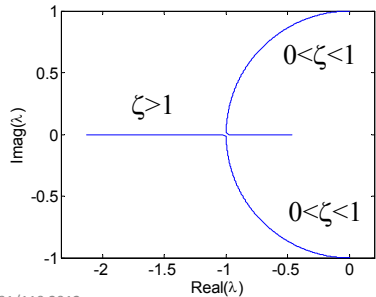
- If you understand response of first and second order systems, you understand the response for any order (eigenvalues of A are either real or complex)
 - Exception is non-diagonalizable A (non-trivial Jordan form)

$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2q = u \quad \leftrightarrow$$


$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$


For $\zeta < 1$, eigenvalues at

$$(-\zeta \pm j\sqrt{1-\zeta^2})\omega_0$$



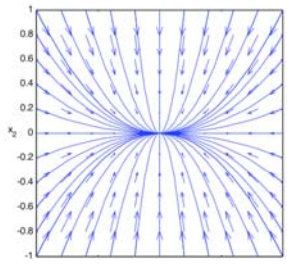
- Analytical formulas exist for overshoot, rise time, settling time, etc

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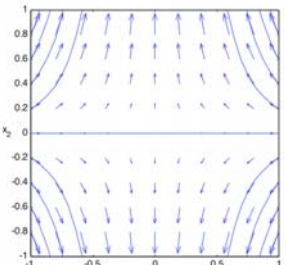


Eigenstructure of Linear Systems

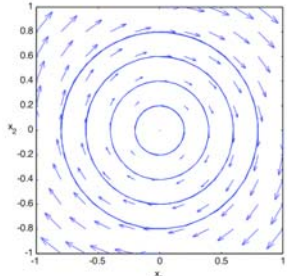
Real e-values
 $\text{Re}(\lambda_i) < 0$



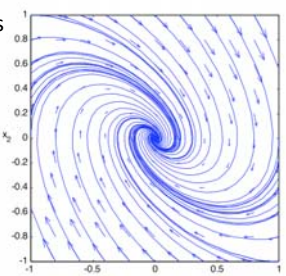
Real e-values
 $\text{Re}(\lambda_i) < 0$
 $\text{Re}(\lambda_i) > 0$




Complex e-values
 $\text{Re}(\lambda_i) = 0$



Complex e-values
 $\text{Re}(\lambda_i) < 0$

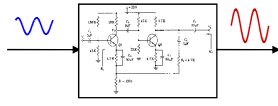


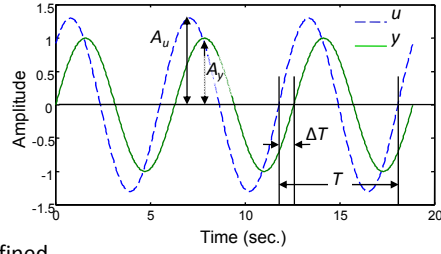
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Frequency Response

- Measure the *steady state* response of the system to sinusoidal input
 - Example: audio amplifier – would like consistent (“flat”) amplification between 20 Hz & 20,000 Hz
 - Individual sinusoids are good *test signals* for measuring performance in many systems (e.g., seasonal cycles in temperature)
- Approach: plot input and output, measure *relative* amplitude and phase
 - Use MATLAB or SIMULINK to generate response of system to sinusoidal output
 - Gain = A_y/A_u
 - Phase = $2\pi \cdot \Delta T/T$
- May not work for *nonlinear* systems
 - System nonlinearities can cause *harmonics* to appear in the output
 - Amplitude and phase may not be well-defined
 - For *linear* systems, frequency response is always well defined



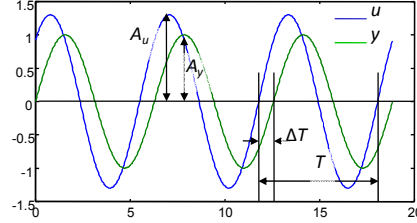


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Computing Frequency Responses

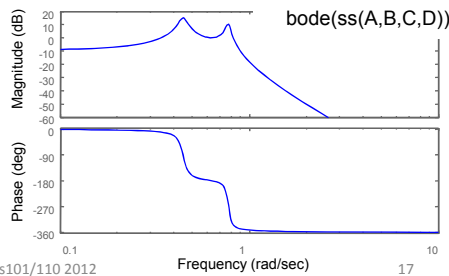
- Technique #1: plot input and output, measure relative amplitude and phase
 - Generate response of system to sinusoidal output
 - Gain = A_y/A_u
 - Phase = $2\pi \cdot \Delta T/T$
 - For *linear* system, gain and phase don't depend on the input amplitude




- Technique #2 (linear systems): use bode (or freqresp) command
 - Assumes linear dynamics in state space form:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$
 - Gain plotted on log-log scale
 - dB = $20 \log_{10}(\text{gain})$
 - Phase plotted on linear-log scale



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Calculating Frequency Response from convolution equation... (more later)

- Convolution equation describes response to any input; use this to look at response to sinusoidal input: $u(t) = A \sin(\omega t) = \frac{A}{2i} (e^{i\omega t} - e^{-i\omega t})$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B e^{i\omega\tau} d\tau$$

$$= e^{At}x(0) + e^{At} \int_0^t e^{(i\omega I - A)\tau} B d\tau$$

$$= e^{At}x(0) + e^{At}(i\omega I - A)^{-1} e^{(i\omega I - A)\tau} \Big|_{\tau=0}^t B$$


$$= e^{At}x(0) + e^{At}(i\omega I - A)^{-1} (e^{(i\omega I - A)t} - I) B$$

$$= \underbrace{e^{At} (x(0) - (i\omega I - A)^{-1} B)}_{\text{Transient (decays if stable)}} + \underbrace{(i\omega I - A)^{-1} B e^{i\omega t}}_{\text{Ratio of response/input}}$$

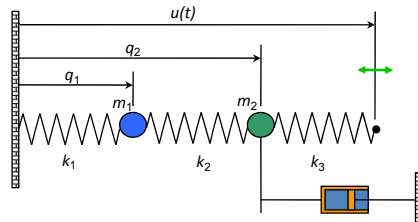
$$y(t) = Cx(t) + Du(t)$$

$$= C e^{At} (x(0) - (i\omega I - A)^{-1} B) + \underbrace{(C(i\omega I - A)^{-1} B + D)}_{\text{"Frequency response"}} e^{i\omega t}$$

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Spring Mass System



$$\frac{d}{dt} \begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m} & \frac{k_2}{m} & 0 & 0 \\ \frac{k_2}{m} & -\frac{k_2+k_3}{m} & 0 & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

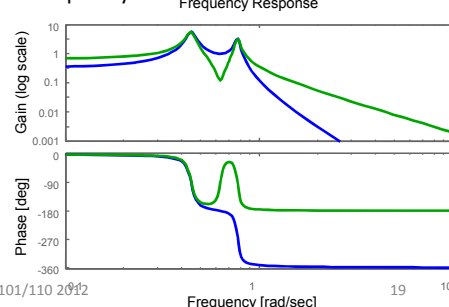
With $k_1 = k_2 = 1, m = 1, c = 0$

$$v_{1,2} = \begin{bmatrix} 1 \\ 1 \\ \pm 1i \\ \pm 1i \end{bmatrix} \quad v_{3,4} = \begin{bmatrix} 1 \\ -1 \\ \pm\sqrt{2}i \\ \mp\sqrt{2}i \end{bmatrix}$$

Eigenvalues of A:


- For zero damping, $\pm j\omega_1$ and $\pm j\omega_2$
- ω_1 and ω_2 correspond to the two peaks in the frequency response
- The eigenvectors for these eigenvalues give the *mode shape*:
 - In-phase motion for the lower frequency
 - Out-of phase motion for the higher frequency

Frequency Response



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Frequency Response of the Climate

- Sinusoidal variation of radiative forcing, evaluate frequency response
- One- or two-reservoir energy balance models do a poor job, e.g.

$$C \frac{dT}{dt} = F - \lambda T \quad \Rightarrow \quad H_1(s) = \frac{1}{Cs + \lambda}$$

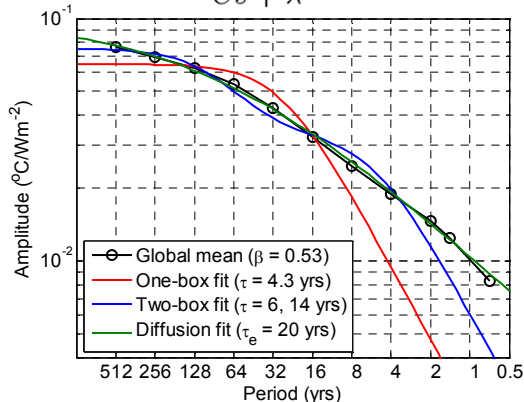
- Global mean temperature response consistent with diffusion into semi-infinite medium:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2}$$

$$\kappa \frac{\partial T}{\partial z} \Big|_{z=0} = F - \lambda T(t, 0)$$

$$\Downarrow$$

$$H_D(s) = \frac{1}{\lambda + \kappa(s/\alpha)^{1/2}}$$




Legend:
 -○- Global mean ($\beta = 0.53$)
 - One-box fit ($\tau = 4.3$ yrs)
 - Two-box fit ($\tau = 6, 14$ yrs)
 - Diffusion fit ($\tau_e = 20$ yrs)

MacMynowski, Shin, Caldeira, *Geoph. Res. Lett.*, 2011.
<http://www.youtube.com/watch?v=Pjv4zL1HR70>

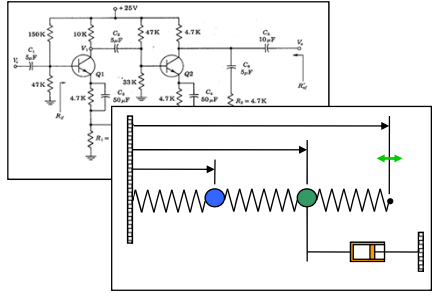
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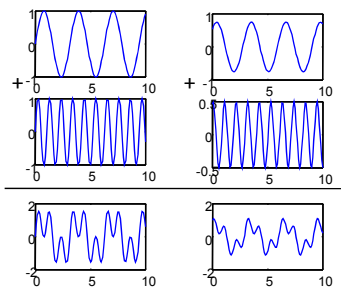
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Summary: Linear Systems

$$\begin{aligned}
 u &\rightarrow \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \\ x(0) = 0 \end{cases} \rightarrow y
 \end{aligned}$$





$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

- Properties of linear systems
 - Linearity with respect to initial condition and inputs
 - Stability characterized by eigenvalues
 - Many applications and tools available
 - Provide local description for nonlinear systems

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