CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 101

Problem Set #1

D. MacMynowski — Issued: 26 Sep 11
Fall 2011 — Due: 3 Oct 11

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. Åström and Murray, Exercise 1.2

2. Consider the cruise-control example discussed in class, with

$$m\dot{v} = -av + u + w$$

where u is the control input (force applied by engine) and w the disturbance input (force applied by hill, etc.), which will be ignored below ($w = 0$). An *open-loop* control strategy to achieve a given reference speed v_{ref} would be to choose

$$u = \hat{a}v_{\text{ref}}$$

where $\hat{a}$ is your estimate of a, which may not be accurate.

(a) Compute the steady-state response for both the open-loop strategy above, and for the feedback law

$$u = -k_p(v - v_{\text{ref}})$$

and compare the steady-state (with $w = 0$) as a function of $\beta = a/\hat{a}$ when $k_p = 10\hat{a}$. (You should solve the problem analytically, and then plot the response $v_{\text{ss}}/v_{\text{ref}}$ as a function of β.)

(b) Now consider a proportional-integral control law

$$u = -k_p(v - v_{\text{ref}}) - k_i \int_0^t (v - v_{\text{ref}})dt$$

and again compute the steady state solution (assuming stability) and compare the response with the proportional gain case from above. (Note that if you define $q = \int_0^t (v - v_{\text{ref}})dt$ then $\dot{q} = v - v_{\text{ref}}$.)

3. Åström and Murray, Exercise 2.6, parts (a) and (b)

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CDS 110a

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2. Åström and Murray, Exercise 2.1

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4. Consider a damped spring–mass system with dynamics

$$m\ddot{q} + c\dot{q} + kq = F.$$

Let $\omega_0 = \sqrt{k/m}$ be the natural frequency and $\zeta = c/(2\sqrt{km})$ be the damping ratio.

(a) Show that by rescaling the equations, we can write the dynamics in the form

$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = \omega_0^2 u, \tag{S1.1}$$

where $u = F/k$. This form of the dynamics is that of a linear oscillator with natural frequency ω_0 and damping ratio ζ.

(b) Show that the system can be further normalized and written in the form

$$\frac{dz_1}{d\tau} = z_2, \qquad \frac{dz_2}{d\tau} = -z_1 - 2\zeta z_2 + v. \tag{S1.2}$$

The essential dynamics of the system are governed by a single damping parameter ζ. The *Q-value* defined as $Q = 1/2\zeta$ is sometimes used instead of ζ.

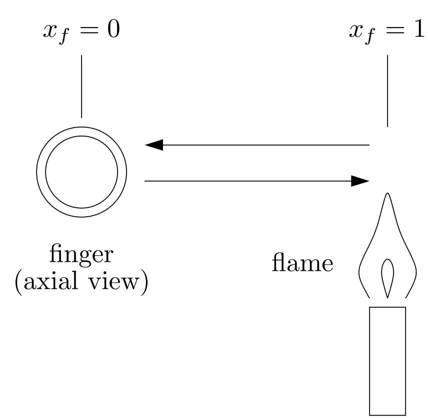
(c) Show that the solution for the unforced system ($v = 0$) with no damping ($\zeta = 0$) is given by

$$z_1(\tau) = z_1(0)\cos\tau + z_2(0)\sin\tau, \qquad z_2(\tau) = -z_1(0)\sin\tau + z_2(0)\cos\tau.$$

Invert the scaling relations to find the form of the solution $q(t)$ in terms of $q(0)$, $\dot{q}(0)$ and ω_0.

(d) Consider the case where $\zeta = 0$ and $u(t) = \sin\omega t$, $\omega > \omega_0$. Solve for $z_1(\tau)$, the normalized output of the oscillator, with initial conditions $z_1(0) = z_2(0) = 0$ and use this result to find the solution for $q(t)$.

5. [Contributed by D. Spanos, 2004; M. Dunlop, 2006] In this problem we will look at how to play with fire without getting burned. The system we want to consider is a finger being moved back and forth across a flame as shown below.



The description of the system is as follows:

- The temperature of a finger is regulated by an internal feedback mechanism. To first order, we will say that heat is convected away by blood flow, at a rate

$$F_b = \alpha_b(T_f - T_b)$$

where T_f is the temperature of the fingertip, T_b is the temperature of the blood, and α_b is the convection coefficient (the F signifies the heat flux).

- A flame gives off heat into the ambient air, and we assume a steady-state temperature field around the flame. The ambient air far from the flame is at 25 degrees Celsius.
- The flame is fixed at $x_F = 1$, and fingertip begins at a position $x_f = 0$, where the ambient air is precisely at the same temperature as the blood.
- Suppose that the temperature of the air varies exponentially with distance from the flame, so

$$T_a(x) = 25 + (T_F - 25)\left(\frac{T_b - 25}{T_F - 25}\right)^{(x-1)^2}$$

  where T_F is the flame temperature.
- Heat convects into the finger from the ambient air at a rate

$$F_a = \alpha_a(T_a - T_f).$$

- The dynamics of the fingertip temperature is given by

$$c_f \frac{T_f}{dt} = -F_b + F_a$$

  where c_f is the fingertip thermal capacity.
- The fingertip is rapidly passed into and out of the flame, according to

$$x_f(t) = \sin(\omega t).$$

Using the MATLAB `ode45` function (or something similar), build a model for the system and solve the following:

(a) Assume that the finger moves sinusoidally in and out of the flame at frequency $\omega = 1$ rad/s. Plot the temperature of the finger as a function of time. After an initial transient (which may not be obvious in the plot), the response should be periodic; what is the "steady-state" peak-to-peak value of the response?

(b) Plot the *steady-state* amplitude (defined above) of the finger temperature as a function of the frequency ω for ω ranging from 1 to 100 rad/s. You should compute at least 5 points in your graph.

(c) Double the "gain" of the temperature control system by increasing α_b by a factor of 2. Replot the frequency response from part 5b and describe in words how it differs from the original gain (i.e., where is the response bigger, smaller or unchanged and what is the reason).

You should use the following parameter values in your simulations:

- $T_b = 37$, $T_F = 1400$ degrees Celsius.
- $\frac{\alpha_a}{c_f} = 1\ s^{-1}$
- $\frac{\alpha_b}{c_f} = 20, 40\ s^{-1}$