Notes on the Laplace Transform

The *one-sided* *Laplace transform* of a signal x(t) is defined by

$$X(s) \triangleq \mathcal{L}_s\{x\} \triangleq \int_0^\infty x(t)e^{-st}dt$$

where t is real and $s = \sigma + j\omega$ is a complex variable. The one-sided Laplace transform is also called the *unilateral* Laplace transform. There is also a *two-sided*, or *bilateral*, Laplace transform obtained by setting the lower integration limit to $-\infty$ instead of 0. Since we will be analyzing only *causal* linear systems using the Laplace transform, we can use either. However, it is customary in engineering treatments to use the one-sided definition.

When evaluated along the $s = j\omega$ axis (*i.e.*, $\sigma = 0$), the Laplace transform reduces to the unilateral *Fourier transform*:

$$X(j\omega) = \int_0^\infty x(t)e^{-j\omega t}dt.$$

The Fourier transform is normally defined bilaterally ($0 \leftarrow -\infty$ in the above integral), but for causal signals there is no difference. We see that the Laplace transform can be viewed as a generalization of the Fourier transform from the real line (a simple frequency axis) to the entire complex plane. We say that the Fourier transform is obtained by evaluating the Laplace transform along the $j\omega$ axis in the complex s plane.

An advantage of the Laplace transform is the ability to transform signals which have no Fourier transform. To see this, we can write the Laplace transform as

$$X(s) = \int_0^\infty x(t)e^{-(\sigma+j\omega)t}dt = \int_0^\infty \left[x(t)e^{-\sigma t}\right]e^{-j\omega t}dt.$$

Thus, the Laplace transform can be seen as the Fourier transform of an *exponentially windowed* input signal. For $\sigma > 0$ (the so-called ``*strict right-half plane*'' (RHP)), this exponential weighting forces the Fourier-transformed signal toward zero as $t \to \infty$. As long as the signal $x(t)$ does not increase faster than *exp*(Bt) for some B, its Laplace transform will exist for all $\sigma > B$.

Existence of the Laplace Transform

A function *x*(t) has a Laplace transform whenever it is of *exponential order*. That is, there must be a real number B such that

$$\lim_{t\to\infty}\left|x(t)e^{-Bt}\right| = 0$$

As an example, every exponential function $Ae^{\alpha t}$ has a Laplace transform for all finite values of A and α. Let's look at this case more closely.

The Laplace transform of a causal, growing exponential function

$$x(t) = \begin{cases} Ae^{\alpha t}, & t \geq 0 \\ 0, & t < 0 \end{cases},$$

is given by

$$\begin{aligned} X(s) &\triangleq \int_0^\infty x(t)e^{-st}dt = \int_0^\infty Ae^{\alpha t}e^{-st}dt = A\int_0^\infty e^{(\alpha-s)t}dt \\ &= \frac{A}{\alpha-s}e^{(\alpha-s)t}\Big|_0^\infty = \frac{A}{\alpha-s}e^{(\alpha-\sigma-j\omega)\infty} - \frac{A}{\alpha-s} \\ &= \begin{cases} \frac{A}{s-\alpha}, & \sigma > \alpha \\ \text{(indeterminate)}, & \sigma = \alpha \\ \infty, & \sigma < \alpha \end{cases} \end{aligned}$$

Thus, the Laplace transform of an exponential $Ae^{\alpha t}$ is $A/(s - \alpha)$, but this is defined only for re$\{s\} > \alpha$

Analytic Continuation

The domain of definition of the Laplace transform can be extended by means of *analytic continuation*. Analytic continuation is carried out by expanding a function of a complex variable *s* about all points in its domain of definition, and extending the domain of definition to all points for which the series expansion converges.

In the case of our exponential example

$$X(s) = \frac{A}{\alpha - s}, \quad (\tag{D.1}$$

the Taylor series expansion of X(s) about the point $s = s_0$ in the s plane is given by

$$\begin{aligned} X(s) &= X(s_0) + (s - s_0)X'(s_0) + (s - s_0)^2 \frac{X''(s_0)}{2} + (s - s_0)^3 \frac{X'''(s_0)}{3!} + \cdots \\ &\triangleq \sum_{n=0}^{\infty} (s - s_0)^n \frac{X^{(n)}(s_0)}{n!} \end{aligned}$$

where, writing $X(s)$ as $(\alpha - s)^{-1}$ and using the chain rule for differentiation,

$$\begin{aligned} X'(s_0) &\triangleq X^{(1)}(s_0) \triangleq \left.\frac{dX(s)}{ds}\right|_{s=s_0} = (-1)(\alpha - s)^{-2}(-1)\big|_{s=s_0} = \frac{1}{(\alpha - s)^2} \\ X''(s_0) &\triangleq X^{(2)}(s_0) \triangleq \left.\frac{d^2X(s)}{ds^2}\right|_{s=s_0} = (-2)(\alpha - s)^{-3}(-1)\big|_{s=s_0} = \frac{2}{(\alpha - s)^3} \\ X'''(s_0) &\triangleq X^{(3)}(s_0) \triangleq \left.\frac{d^3X(s)}{ds^3}\right|_{s=s_0} = (-3)(2)(\alpha - s)^{-4}(-1)\big|_{s=s_0} = \frac{3!}{(\alpha - s)^4} \end{aligned}$$

and so on. We also used the *factorial notation*

$$n! \triangleq n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$$

and we defined the special cases $0! \triangleq 1$ and $X^{(0)}(s_0) = X(s_0)$, as is normally done. The series expansion of X(s) can thus be written

$$\begin{aligned} X(s) &= \frac{1}{\alpha - s_0} + \frac{s - s_0}{(\alpha - s_0)^2} + \frac{(s - s_0)^2}{(\alpha - s_0)^3} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(s - s_0)^n}{(\alpha - s_0)^{n+1}}. \end{aligned} \tag{D.2}$$

We now ask for what values of s does the series Eq. (D.2) *converge*? The value $s = \alpha$ is particularly easy to check, since

$$X(\alpha) = \sum_{n=0}^{\infty} \frac{(\alpha - s_0)^n}{(\alpha - s_0)^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\alpha - s_0} = \infty \frac{1}{\alpha - s_0}.$$

Thus, the series clearly does *not* converge for $s = \alpha$, no matter what our choice of s_0 might be. We must therefore accept the point at infinity for $H(\alpha)$. This is eminently reasonable since the closed form Laplace transform we derived, $H(s) = 1/(\alpha - s)$ *does* ``blow up'' at $s = \alpha$. The point $s = \alpha$ is called a *pole* of $H(s) = 1/(\alpha - s)$.

More generally, let's apply the *ratio test* for the convergence of a geometric series. Since the nth term of the series is

$$\frac{(s - s_0)^n}{(\alpha - s_0)^{n+1}}$$

the ratio test demands that the ratio of term n+1 over term n have absolute value less than 1. That is, we require

$$1 > \left| \frac{(s - s_0)^{n+1}}{(\alpha - s_0)^{n+2}} \bigg/ \frac{(s - s_0)^n}{(\alpha - s_0)^{n+1}} \right| = \left| \frac{s - s_0}{\alpha - s_0} \right|,$$

or,

$$\boxed{|s - s_0| < |\alpha - s_0| \, .}$$

We see that the region of convergence is a circle about the point $s = s_0$ having radius approaching but not equal to $|\alpha - s_0|$. Thus, the circular disk of convergence is centered at $s = s_0$ and extends to, but does not touch, the *pole* at $s = \alpha$.

The *analytic continuation* of the domain of Eq. (D.1) is now defined as the *union* of the disks of convergence for all points $s_0 \neq \alpha$. It is easy to see that a sequence of such disks can be chosen so as to define all points in the s plane except at the pole $s = \alpha$.

In summary, the Laplace transform of an exponential $x(t) = Ae^{\alpha t}$

$$X(s) = \frac{A}{s - \alpha}$$

and the value is well defined and finite for all $s \neq \alpha$.

Analytic continuation works for any finite number of poles of finite order, and for an infinite number of distinct poles of finite order. It breaks down only in pathological situations such as when the Laplace transform is singular everywhere on some closed contour in the complex plane. Such pathologies do not arise in practice, so we need not be concerned about them.