CDS 101/110a: Lecture 7-1
Loop Analysis of Feedback Systems

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Goals:

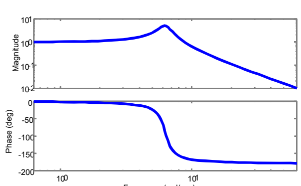
- Compute closed loop stability from open loop properties:
  - Nyquist stability criterion for stability of feedback systems
- Define gain and phase margin and determine it from Nyquist and Bode plots

Reading:

- Åström and Murray, *Feedback Systems,* Ch 9.1-9.4
- *Advanced*: Lewis, Chapters 7

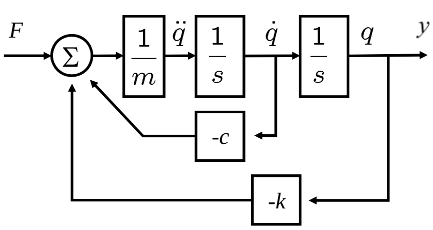
Review From Last Week

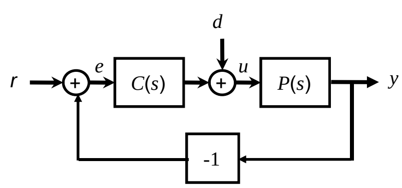
$$u = A\sin(\omega t) \rightarrow \begin{array}{c} \dot{x} = Ax + Bu \\ y = Cx + Du \\ x(0) = 0 \end{array} \rightarrow y_{ss} = A \cdot |G(i\omega)| \times \sin(\omega t + \arg G(i\omega))$$



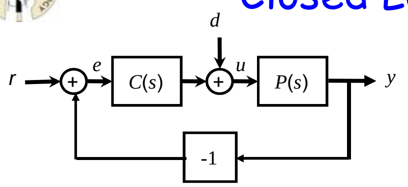
$$G(s) = C(sI - A)^{-1}B + D$$

$$G_{y_2u_1} = G_{y_2u_2}G_{y_1u_1} = \frac{n_1n_2}{d_1d_2}$$





Closed Loop Stability



Q: how do open loop dynamics affect the closed loop stability?

– Given open loop transfer function $C(s)P(s)$ determine when system is stable

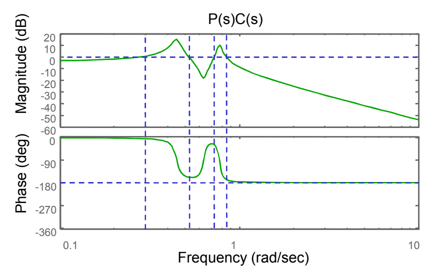
Brute force answer: compute poles of closed loop transfer function

$$H_{yr} = \frac{PC}{1+PC} = \frac{n_p n_c}{d_p d_c + n_p n_c}$$

- Poles of H_{yr} = zeros of 1 + PC
- Easy to compute, but not so good for design

Alternative: look for conditions on *PC* that lead to instability

- Example: if $PC(s) = -1$ for some $s = i\omega$, then system is *not* asymptotically stable
- Condition on *PC* is much nicer because we can *design* $PC(s)$ by choice of $C(s)$
- However, checking $PC(s) = -1$ is not enough; need more sophisticated check

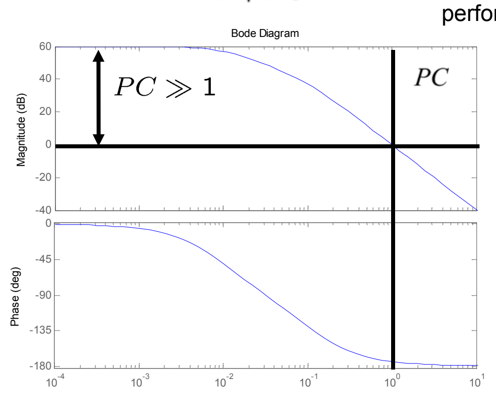


Game Plan: Frequency Domain Design

- Goal: figure out how to *design* $C(s)$ so that $1+C(s)P(s)$ is stable *and* we get good performance

$$H_{yr} = \frac{PC}{1+PC}$$

- Poles of H_{yr} = zeros of 1 + *PC*
- Would also like to "shape" H_{yr} to specify performance at different frequencies



- Low frequency range:

$$PC \gg 1 \;\Rightarrow\; \frac{PC}{1+PC} \simeq 1$$

(good tracking)

- Bandwidth: frequency at which closed loop response = $1/\sqrt{2}$ $\Rightarrow$ open loop gain ≈ 1
- Idea: use $C(s)$ to *shape PC* (under certain constraints)
- Need tools to analyze stability and performance for closed loop given *PC*

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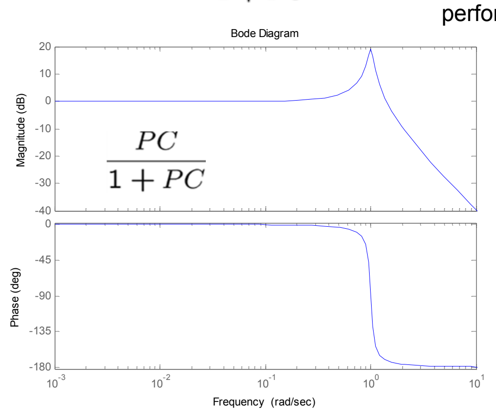

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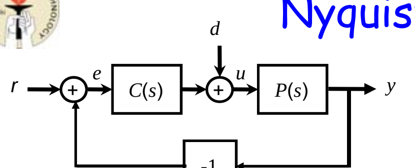
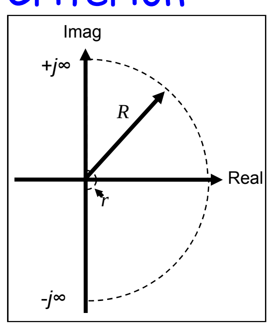
Nyquist Criterion



Determine stability from (open) loop transfer function, $L(s) = P(s)C(s)$.

- Use "principle of the argument" from complex variable theory



- Nyquist "D" contour
- Take limit as $r \to 0$, $R \to \infty$
- Trace from $-\infty$ to $+\infty$ along imaginary axis

Thm (Nyquist). Consider the Nyquist plot for loop transfer function $L(s)$. Let

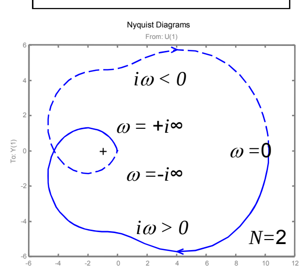
P # RHP poles of $L(s)$

N # clockwise encirclements of -1

Z # RHP zeros of $1 + L(s)$

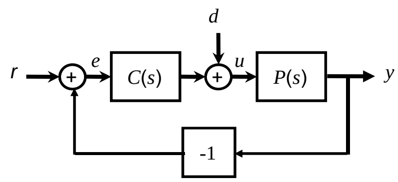
Then

$$Z = N + P$$



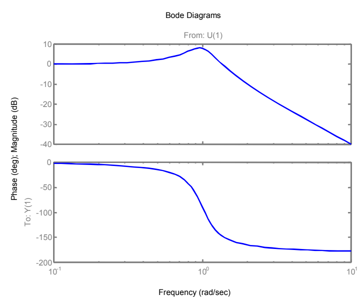
- Trace frequency response for $L(s)$ along the Nyquist "D" contour
- Count net # of clockwise encirclements of the -1 point

Simple Interpretation of Nyquist

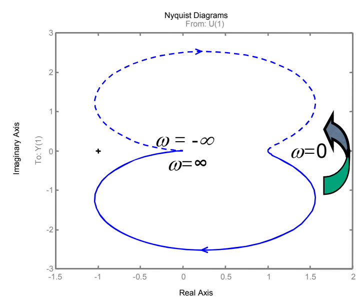


- Basic idea: avoid positive feedback
  - If $L(s)$ has 180˚ phase (or greater) and gain greater than 1, then signals are amplified around loop
  - Use when phase is monotonic
  - General case requires Nyquist

Can generate Nyquist plot from Bode plot + reflection around real axis

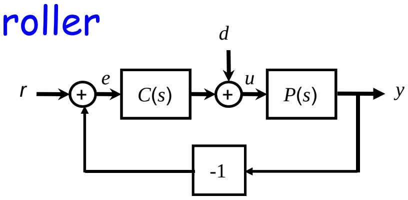


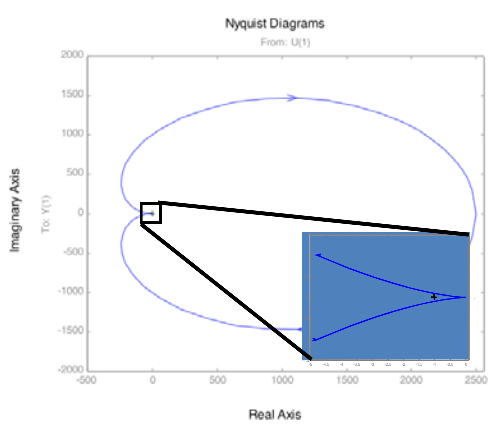
ambode(sys) [or bode(sys) in dB]



amnyquist(sys)

Example: Proportional + Integral* speed controller





$$P(s) = \frac{1/m}{s + b/m} \times \frac{r}{s + a}$$

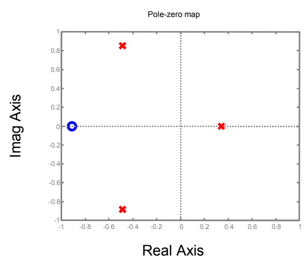
$$C(s) = K_p + \frac{K_i}{s + 0.01}$$

- Remarks
  - $N = 0, P = 0 \Rightarrow Z = 0$ (stable)
  - Need to zoom in to make sure there are no net encirclements
  - Note that we don't have to compute closed loop response

* slightly modified; more on the design of this compensator in next week's lecture

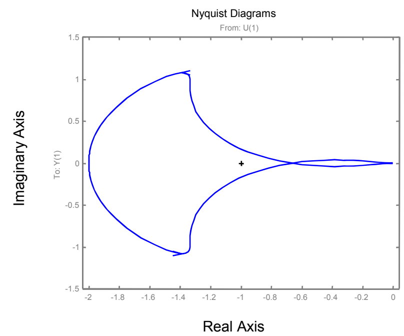
More complicated systems

- What happens when open loop plant has RHP poles?
  - 1 + PC has singularities inside D contour ⇒ these must be taken into account



$$L(s) = \frac{s+1}{s-0.5} \times \frac{1}{s^2+s+1}$$

unstable pole



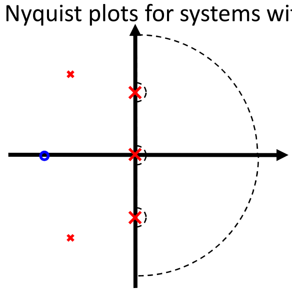
N = -1, P = 1 ⇒ $Z = N+P$ = 0 (stable)

$$\frac{1}{1+L} = \frac{s+1}{(s+0.35)(s+0.07+1.2i)(s+0.07-1.2i)}$$ ✓

Comments and cautions

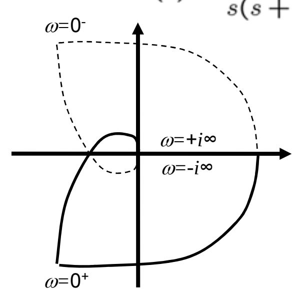
- Why is the Nyquist plot *useful*?
  - Old answer: easy way to compute stability (before computers and MATLAB)
  - Real answer: gives *insight* into stability and robustness; very useful for reasoning about stability

- Nyquist plots for systems with poles on the $j\omega$ axis



- chose contour to avoid poles on axis
- need to carefully compute Nyquist plot at these points
- evaluate $H(\varepsilon+0i)$ to determine direction

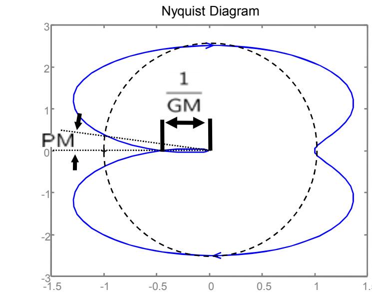
$$H(s) = \frac{1}{s(s+1)^2}$$

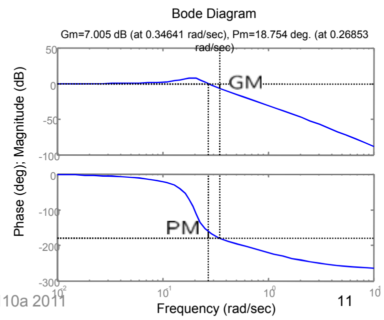


- Cautions with using MATLAB
  - MATLAB doesn't generate portion of plot for poles on imaginary axis
  - These must be drawn in by hand (make sure to get the orientation right!)

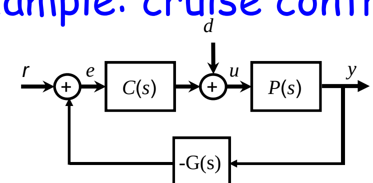
Robust stability: gain and phase margins

- Nyquist plot tells us if closed loop is stable, but not *how* stable
- Gain margin
  - How much we can modify the *loop gain* and still have the system be stable
  - Determined by the location where the loop transfer function crosses 180˚ phase
- Phase margin
  - How much we can add "phase delay" and still have the system be stable
  - Determined by the phase at which the loop transfer function has unity gain
- Bode plot interpretation
  - Look for gain = 1, 180˚ phase crossings
  - MATLAB: margin(sys)





Example: cruise control

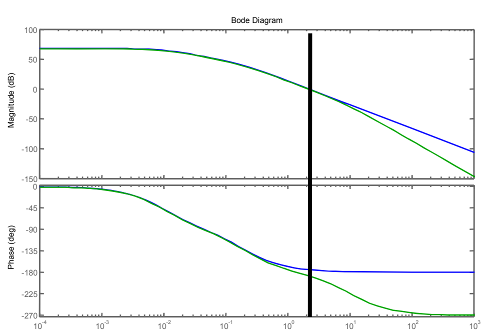


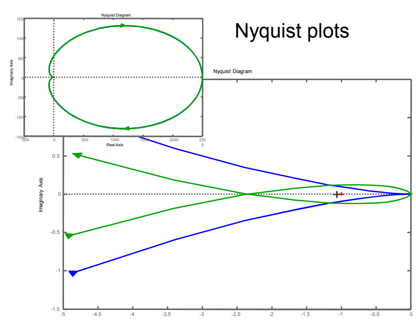
$$P(s) = \frac{1/m}{s + b/m} \times \frac{r}{s+a}$$

$$C(s) = K_p + \frac{K_i}{s + 0.01}$$

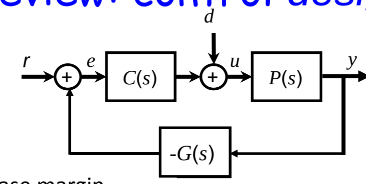
$$G(s) = \frac{10}{s+10}$$

- Effect of additional sensor dynamics
  - New speedometer has pole at s = -10 (very fast); problems develop in the field
  - What's the problem? A: insufficient phase margin in original design (not robust)





Preview: control design

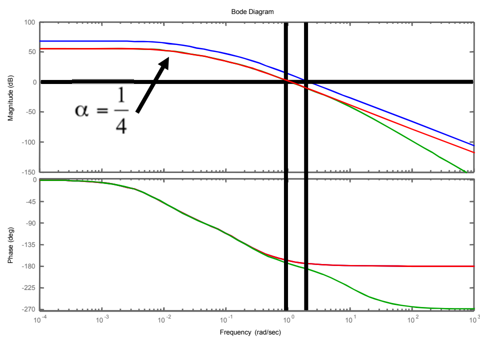


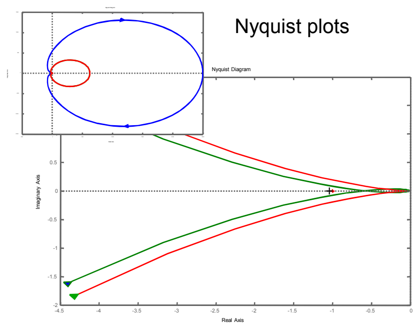
$$P(s) = \frac{1/m}{s + b/m} \times \frac{r}{s+a}$$

$$C(s) = \alpha\left(K_p + \frac{K_i}{s+0.01}\right)$$

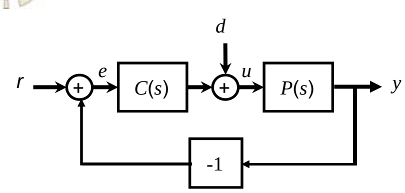
$$G(s) = \frac{10}{s+10}$$

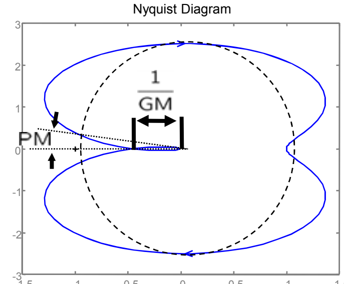
- Approach: Increase phase margin
  - Increase phase margin by reducing gain ⇒ can accommodate new sensor dynamics
  - Tradeoff: lower gain at low frequencies ⇒ less bandwidth, larger steady state error



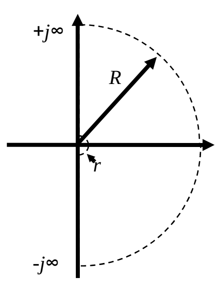


Summary: Loop Analysis of Feedback Systems





- Nyquist criteria for loop stability
- Gain, phase margin for robustness



Thm (Nyquist).

P	# RHP poles of *L*(*s*)
N	# CW encirclements
Z	# RHP zeros

$Z = N + P$