



# CDS 101/110a: Lecture 6-1 Transfer Functions

Douglas G. MacMynowski

**Goals:**

- Motivate and define the input/output transfer function of a linear system
- Understand the relationships among frequency response (Bode plot), transfer function, and state-space model
- Introduce block diagram algebra for transfer functions of interconnected systems

**Reading:**

- Åström and Murray, *Feedback Systems*, Ch 8
- *Advanced*: Lewis, Chapters 3-4 or DFT, Chapter 2

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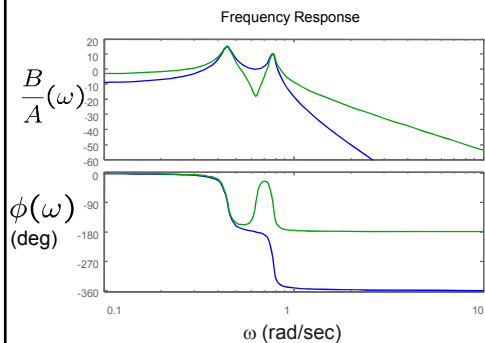
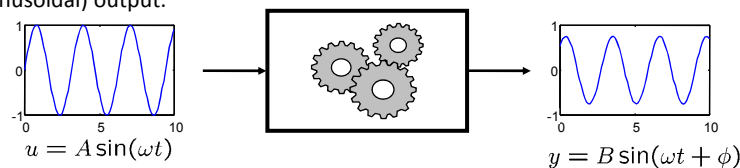
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# Frequency Domain Modeling

**Defn.** The *frequency response* of a linear system is the relationship between the gain and phase of a sinusoidal input and the corresponding steady state (sinusoidal) output.



**Bode plot (1940; Henrik Bode)**

- Plot gain and phase vs input frequency
- Gain is plotting using log-log plot
- Phase is plotting with log-linear plot
- Can read off the system response to a sinusoid – in the lab or in simulations
- Linearity  $\Rightarrow$  can construct response to any input (via Fourier decomposition)
- Key idea: do all computations in terms of gain and phase (frequency domain)

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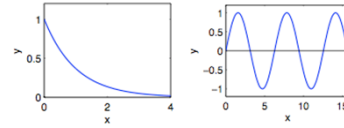
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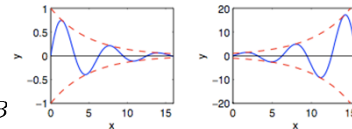
## Transmission of Exponential Signals

- Exponential signal:  $e^{st} = e^{(\sigma+i\omega)t} = e^{\sigma t} e^{i\omega t} = e^{\sigma t} (\cos \omega t + i \sin \omega t)$ 
  - Construct constant inputs + sines/cosines by linear combinations
  - Constant:  $u(t) = c = ce^{0t}$
  - Sinusoid:  $u(t) = A \sin(\omega t) = \frac{A}{2i} (e^{i\omega t} - e^{-i\omega t})$



- Exponential response can be computed via the convolution equation

$$\begin{aligned}
 x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)} B e^{s\tau} d\tau \\
 &= e^{At}x(0) + e^{At}(sI - A)^{-1} e^{(sI-A)\tau} \Big|_{\tau=0}^t B \\
 &= e^{At}x(0) + e^{At}(sI - A)^{-1} (e^{(sI-A)t} - I) B \\
 &= e^{At} (x(0) - (sI - A)^{-1} B) + (sI - A)^{-1} B e^{st} \\
 y(t) &= Cx(t) + Du(t) \\
 &= C e^{At} (x(0) - (sI - A)^{-1} B) + (C(sI - A)^{-1} B + D) e^{st}
 \end{aligned}$$



- Can also motivate using Laplace transform:  $Y(s)/U(s) = C(sI - A)^{-1} B + D$

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## Transfer Function and Frequency Response

- Exponential response of a linear state space system (from convolution)

$$y = \underbrace{C e^{At} (x(0) - (sI - A)^{-1} B)}_{\text{transient}} + \underbrace{(C(sI - A)^{-1} B + D) e^{st}}_{\text{steady state}}$$

- Transfer function
  - Steady state response is proportional to exponential input => look at input/output ratio
  - $G(s) = C(sI - A)^{-1} B + D$  is the *transfer function* between input and output
- Frequency response

$$\begin{aligned}
 u(t) &= A \sin \omega t = \frac{A}{2i} (e^{i\omega t} - e^{-i\omega t}) \\
 y_{ss}(t) &= \frac{A}{2i} (G(i\omega) e^{i\omega t} - G(-i\omega) e^{-i\omega t}) \\
 &= A \cdot \underbrace{|G(i\omega)|}_{\text{gain}} \sin(\omega t + \underbrace{\arg G(i\omega)}_{\text{phase}})
 \end{aligned}$$

### Common transfer functions

$\dot{y} = u$	$\frac{1}{s}$
$y = \dot{u}$	$s$
$\dot{y} + ay = u$	$\frac{1}{s+a}$
$\ddot{y} = u$	$\frac{1}{s^2}$
$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = u$	$\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
$y = k_p u + k_d \dot{u} + k_i \int u$	$k_p + k_d s + \frac{k_i}{s}$
$y(t) = u(t - \tau)$	$e^{-\tau s}$

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## Poles and Zeros

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

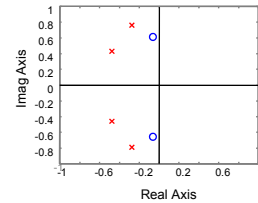
$$H(s) = \frac{n(s)}{d(s)} \quad d(s) = \det(sI - A)$$

- Roots of  $d(s)$  are called *poles* of  $H(s)$
- Roots of  $n(s)$  are called *zeros* of  $H(s)$

- Poles of  $H(s)$  determine the stability of the (closed loop) system
  - Denominator of transfer function = characteristic polynomial of state space system
  - Provides easy method for computing stability of systems
  - Right half plane (RHP) poles ( $\text{Re} > 0$ ) correspond to unstable systems
- Zeros of  $H(s)$  related to frequency ranges with limited transmission
  - A pure imaginary zero at  $s=i\omega_z$  blocks any output at that frequency ( $G(i\omega_z) = 0$ )
  - Zeros provide limits on performance, especially RHP zeros (more on this later)

$$H(s) = k \frac{s^2 + b_1s + b_2}{s^4 + a_1s^3 + a_2s^2 + a_3s + a_4}$$

pzmap



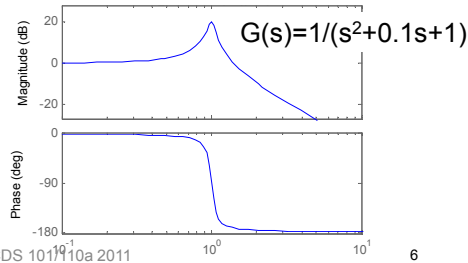
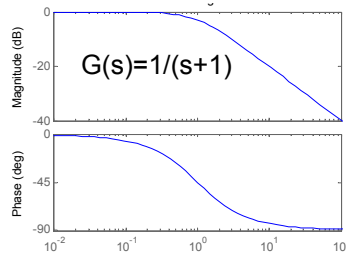
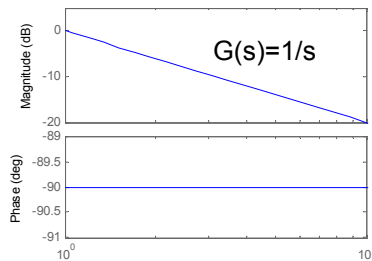
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## Plotting Bode Plots

- Evaluate the transfer function on the imaginary axis
  - This is sufficient to characterize the transfer function, follows from analyticity
- At frequency  $\omega$ , then  $G(i\omega) = r^{i\theta}$




Some useful matlab commands:

```
sys=ss(A,B,C,D);
G=tf(sys);
G=ss2tf(A,B,C,D);
n=[0 0 1];d=[1 0.1 1],G=tf(n,d)
s=tf('s');G=1/(s^2+0.1*s+1);
bode(G)
```

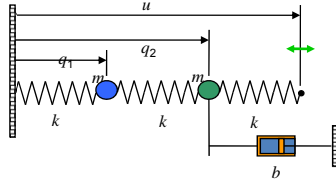
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## Example: Coupled Masses

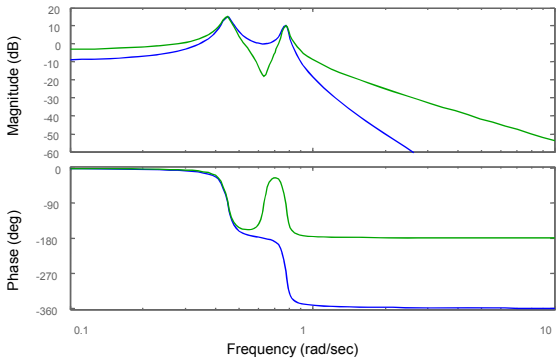


$$H_{q1f}(s) = k \frac{0.04}{s^4 + 0.08s^3 + 0.8016s^2 + 0.032s + 0.12}$$


$$H_{q2f}(s) = k \frac{0.2s^2 + 0.008s + 0.08}{s^4 + 0.08s^3 + 0.8016s^2 + 0.032s + 0.12}$$

- Poles ( $H_{q1f}$  and  $H_{q2f}$ )
  - $-0.0200 \pm 0.7743j$
  - $-0.0200 \pm 0.4468j$
- Zeros ( $H_{q2f}$ )
  - $-0.0200 \pm 0.6321j$
- Interpretation
  - Zeros in  $H_{q2f}$  give low response at  $\omega \simeq 0.6321$

Frequency Response



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## Series Interconnections

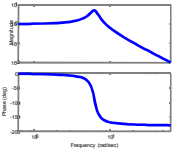
**Q: what happens when we connect two systems together *in series*?**

$$\begin{aligned} \dot{x}_1 &= A_1x_1 + B_1u_1 \\ y_1 &= C_1x_1 + D_1u_1 \end{aligned}$$

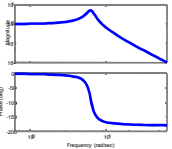
 $\xrightarrow{y_1 = u_2}$ 

$$\begin{aligned} \dot{x}_2 &= A_2x_2 + B_2u_2 \\ y_2 &= C_2x_2 + D_2u_2 \end{aligned}$$


 $\xrightarrow{y_2}$

$u = A \sin(\omega t)$ 


 $\xrightarrow{y = g_1 A \sin(\omega t + \phi_1)}$ 

$y = g_1 A \sin(\omega t + \phi_1)$ 


 $\xrightarrow{y_2 = g_1 g_2 A \sin(\omega t + \phi_1 + \phi_2)}$ 

$y_2 = g_1 g_2 A \sin(\omega t + \phi_1 + \phi_2)$ 


- A: Transfer functions *multiply*
  - Gains multiply
  - Phases add
  - Generally: transfer functions well formulated for frequency domain interconnections

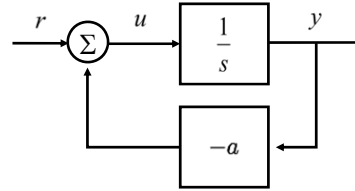
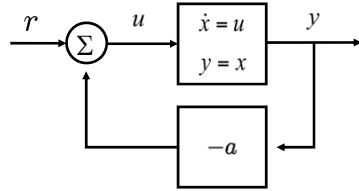
$$u_1 \rightarrow \boxed{G_1(s)} \xrightarrow{u_2 = y_1} \boxed{G_2(s)} \rightarrow y_2$$

$$u_1 \rightarrow \boxed{G_2(s)G_1(s)} \xrightarrow{y_2 = G_2(s)G_1(s)u_1}$$

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## Feedback Interconnection



- State space derivation

$$\begin{aligned}\dot{x} &= u = r - ay = -ax + r \\ y &= x\end{aligned}$$

- Frequency response:  $r = A \sin(\omega t)$

$$y = \left| \frac{1}{\sqrt{a^2 + \omega^2}} \right| \sin\left(\omega t - \tan^{-1}\left(\frac{\omega}{a}\right)\right)$$

- Transfer function derivation

$$\begin{aligned}y &= \frac{u}{s} = \frac{r - ay}{s} \\ y &= \frac{r}{s + a} = G(s)r\end{aligned}$$

- Frequency response

$$y = |G(i\omega)| \sin(\omega t + \angle G(i\omega))$$

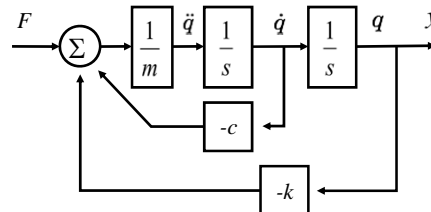
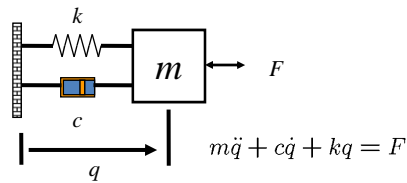
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## Example: mass spring system



- Rewrite in terms of "block diagram"

- Represent integration using  $1/s$
- Include spring and damping through feedback terms

$$y = \frac{1}{m} \cdot \frac{1}{s} \cdot \frac{1}{s} (F - c\dot{q} - kq) = \frac{1}{ms^2} F - \frac{c}{ms} y - \frac{k}{ms^2} y$$

- Determine the transfer function through algebraic manipulation
- Claim: resulting transfer function captures the frequency response

$$\left(1 + \frac{b}{ms} + \frac{k}{ms^2}\right) y = \frac{1}{ms^2} F$$

$$y = \frac{1}{ms^2 + cs + k} F$$

$$H(s) = \frac{1}{ms^2 + cs + k}$$

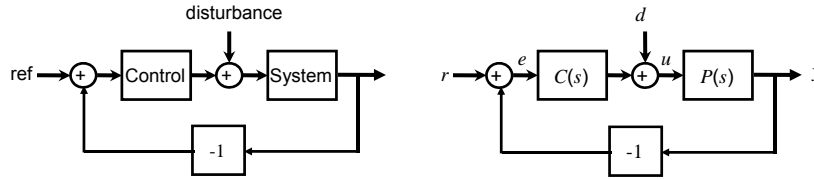
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## Control Analysis and Design Using Transfer Functions



- Transfer functions provide a method for “block diagram algebra”
  - Easy to compute transfer functions between various inputs and outputs
    - $H_{er}(s)$  is the transfer function between the reference and the error
    - $H_{ed}(s)$  is the transfer function between the disturbance and the error
- Transfer functions provide a method for performance specification
  - Since transfer functions provide frequency response directly, it is convenient to work in the “frequency domain”
    - $H_{er}(s)$  should be small in the frequency range 0 to 10 Hz (good tracking)

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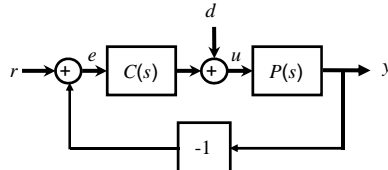
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## Block Diagram Algebra

- Basic idea: treat transfer functions as multiplication, write down equations



$$\begin{aligned}
 y &= P(s)u \\
 u &= d + C(s)e \\
 e &= r - y
 \end{aligned}$$

- Manipulate equations to compute desired signals

$$\begin{aligned}
 e &= r - y \\
 &= r - P(s)u \\
 &= r - P(s)(d + C(s)e) \\
 (1 + P(s)C(s))e &= r - P(s)d \\
 e &= \underbrace{\frac{1}{1 + P(s)C(s)}}_{H_{er}} r - \underbrace{\frac{P(s)}{1 + P(s)C(s)}}_{H_{ed}} d
 \end{aligned}$$

Note: linearity gives superposition of terms

- Algebra works because we are working in frequency domain
  - Time domain (ODE) representations are not as easy to work with
  - Formally, all of this works because of Laplace transforms

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## Block Diagram Algebra

Type	Diagram	Transfer function
Series		$H_{y_2u_1} = H_{y_2u_2} H_{y_1u_1} = \frac{n_1 n_2}{d_1 d_2}$
Parallel		$H_{y_3u_1} = H_{y_2u_1} + H_{y_1u_1} = \frac{n_1 d_2 + n_2 d_1}{d_1 d_2}$
Feedback		$H_{y_1r} = \frac{H_{y_1u_1}}{1 + H_{y_1u_1} H_{y_2u_2}} = \frac{n_1 d_2}{n_1 n_2 + d_1 d_2}$

- These are the basic manipulations needed; some others are possible
- Formally, could work all of this out using the original ODEs (nothing *really* new)

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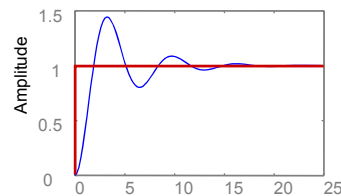
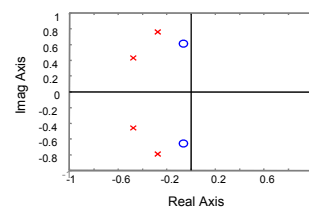


## MATLAB manipulation of transfer functions

- Creating transfer functions
  - [num, den] = ss2tf(A, B, C, D)
  - sys = tf(num, den) or tf(ss(A,B,C,D))
  - num=1, den = [1 a b] →  $s^2 + as + b$
- Interconnecting blocks
  - sys= series(sys1, sys2), parallel, feedback
- Computing poles and zeros
  - pole(sys), zero(sys)
  - pzmap(sys)
- I/O response
  - step(sys), bode(sys)

```

>> tf(sys)
Transfer function:
      1
-----
s^2 + 0.2 s + 1
  
```



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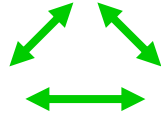
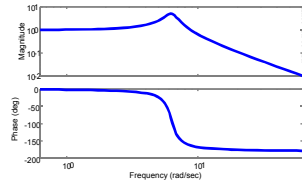
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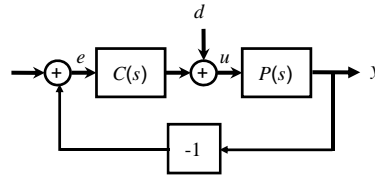
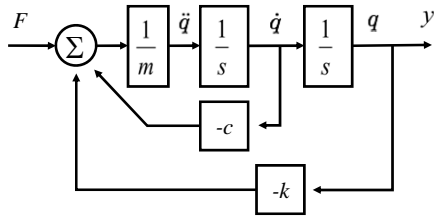
## Summary: Frequency Response & Transfer Functions

$$u = A \sin(\omega t) \rightarrow \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \\ x(0) = 0 \end{cases} \rightarrow y_{ss} = A \cdot |G(i\omega)| \times \sin(\omega t + \arg G(i\omega))$$



$$G(s) = C(sI - A)^{-1}B + D$$

$$G_{y_2 u_1} = G_{y_2 u_2} G_{y_1 u_1} = \frac{n_1 n_2}{d_1 d_2}$$



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