CDS 110a: Lecture 5-2
Laplace Transform & Frequency Response

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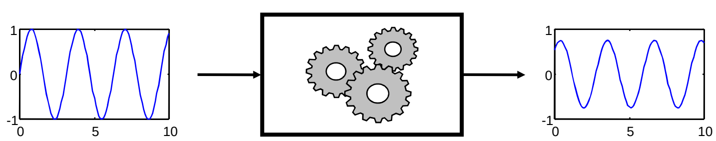
Goals:
- Motivate the role of the Laplace Transform in Control Analysis
- Define the Laplace Transform and its Properties
- Use the Laplace Transform to analyze system response

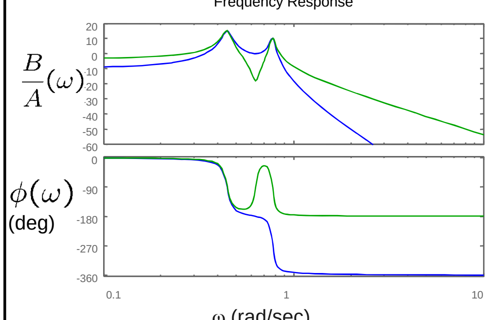
Reading:
- Åström and Murray, *Feedback Systems,* Chapter 8 (Sections 8.1, 8.2)

Frequency Domain Modeling (Chapter 8)

Defn. The *frequency response* of a linear system is the relationship between the gain and phase of a sinusoidal input and the corresponding steady state (sinusoidal) output.



$u = A\sin(\omega t)$ $\qquad$ $y = B\sin(\omega t + \phi)$



Bode plot (1940; Henrik Bode)
- Plot gain and phase vs input frequency
- Gain is plotting using log-log plot
- Phase is plotting with log-linear plot
- Can read off the system response to a sinusoid – in the lab or in simulations
- Linearity ⇒ can construct response to any input (via Fourier decomposition)
- Key idea: do all computations in terms of gain and phase (frequency domain)

Complex Exponentials as Eigenfunctions

Complex Exponentials are Eigenfunctions of LTI operators: $\mathcal{H} f = \lambda f$

$$\underbrace{\int_{-\infty}^{\infty} h(t-\tau)}_{\mathcal{H}} \underbrace{ce^{s\tau}}_{f} \, d\tau = \int_{-\infty}^{\infty} h(\tau) ce^{s(t-\tau)} \, d\tau$$
$$= ce^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} \, d\tau$$
$$= \underbrace{ce^{st}}_{f} \underbrace{H(s)}_{\lambda}$$

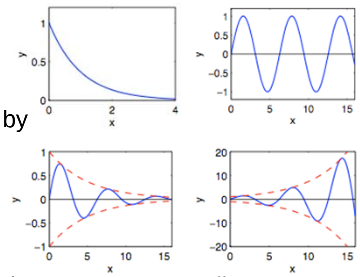
$$\underbrace{H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} \, d\tau}_{\text{Laplace Transform:}}$$

Thus, control inputs/outputs can be written as complex exponentials

Exponential signal:

$$e^{st} = e^{(\sigma + i\omega)t} = e^{\sigma t} e^{i\omega t} = e^{\sigma t}(\cos \omega t + i \sin \omega t)$$

- E.g., construct constant inputs + sines/cosines by linear combinations
  - Constant: $u(t) = c = ce^{0t}$
  - Sinusoid: $u(t) = A \sin(\omega t) = \frac{A}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$



Transmission of Exponential Signals

Exponential signal: $e^{st} = e^{(\sigma + i\omega)t} = e^{\sigma t} e^{i\omega t} = e^{\sigma t}(\cos \omega t + i \sin \omega t)$

- Exponential response can be computed via the convolution equation

$$\begin{aligned} x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)} B e^{s\tau} d\tau \\ &= e^{At}x(0) + e^{At}(sI - A)^{-1} e^{(sI-A)\tau}\Big|_{\tau=0}^{t} B \\ &= e^{At}x(0) + e^{At}(sI - A)^{-1}\left(e^{(sI-A)t} - I\right) B \\ &= e^{At}\left(x(0) - (sI - A)^{-1}B\right) + (sI - A)^{-1} B e^{st} \end{aligned}$$

$$\begin{aligned} y(t) &= Cx(t) + Du(t) \\ &= Ce^{At}\left(x(0) - (sI - A)^{-1}B\right) + \left(C(sI - A)^{-1}B + D\right) e^{st} \end{aligned}$$

Transfer Function and Frequency Response

Exponential response of a linear state space system

$$y = Ce^{At}\left(x(0) - (sI - A)^{-1}B\right) + \left(C(sI - A)^{-1}B + D\right)e^{st}$$

transient — steady state

Transfer function

- Steady state response is proportional to exponential input => look at input/output ratio

$G(s) = C(sI - A)^{-1}B + D$ is the *transfer function* between input and output

Frequency response

$$u(t) = A\sin\omega t = \frac{A}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$$

$$y_{ss}(t) = \frac{A}{2i}\left(G(i\omega)e^{i\omega t} - G(-i\omega)e^{-i\omega t}\right)$$
$$= A \cdot |G(i\omega)| \sin\left(\omega t + \arg G(i\omega)\right)$$

gain — phase

Common transfer functions

$\dot{y} = u$	$\frac{1}{s}$
$y = \dot{u}$	s
$\dot{y} + ay = u$	$\frac{1}{s+a}$
$\ddot{y} = u$	$\frac{1}{s^2}$
$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = u$	$\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
$y = k_p u + k_d \dot{u} + k_i \int u$	$k_p + k_d s + \frac{k_i}{s}$
$y(t) = u(t-\tau)$	$e^{-\tau s}$