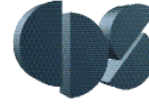




CDS 110a: Lecture 5-1 Observability and State Estimation



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Goals:

- Define observability and give conditions for checking observability for linear control systems
- Introduce the state estimation (aka **observer**) problem
- Provide examples of state estimation in the context of closed loop design
- Start Frequency Domain (Chapter 8)

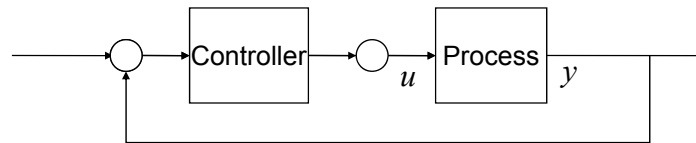
Reading:

- Åström and Murray, *Feedback Systems*, Sections 7.1-7.3

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Questions (from last week)



1) Can the input u affect the dynamics?

$$\dot{x}_1 = x_1 + u$$

e.g. $\dot{x}_2 = x_2 \quad \Rightarrow \text{Can't change } x_2$

Equivalent to asking if there is a u that allows us to reach any state –space point
 \Rightarrow Reachability (*last week*), depends on A, B
 \Rightarrow Related to the design of state feedback

2) Does the measurement y contain enough information about the system?

$$\dot{x}_1 = x_1 \quad y = x_1$$

e.g. $\dot{x}_2 = x_2 \quad \Rightarrow \text{Can't measure } x_2$

\Rightarrow Observability (*this week*), depends on A, C
 \Rightarrow Related to the design of observers to estimate state from measurement

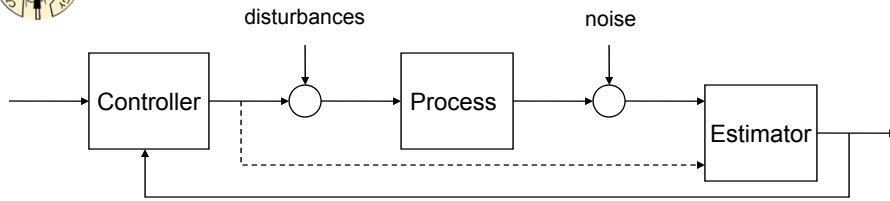
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The State Estimation Problem



Problem Setup

- Given a dynamical system with noise and uncertainty, estimate the state

$$\begin{aligned} \dot{x} &= Ax + Bu + Fv \\ y &= Cx + Du + Gw \end{aligned} \longrightarrow \begin{aligned} \hat{\dot{x}} &= \alpha(\hat{x}, y, u) \leftarrow \text{estimator} \\ \lim_{t \rightarrow \infty} E(x - \hat{x}) &= 0 \end{aligned}$$

\nwarrow expected value

- \hat{x} is called the *estimate* of x

Remarks

- Several sources of uncertainty: noise, disturbances, process, initial condition
 - 110a: Seek estimator that converges in absence of noise: $\lim_{t \rightarrow \infty} \hat{x} = x$
- Uncertainties are unknown, except through their effect on measured output
- First question: when is this even *possible*?

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Observability

Defn A dynamical system of the form

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x, u) \end{aligned}$$

is *observable* if for any $T > 0$ it is possible to determine the state of the system $x(T)$ through measurements of $y(t)$ and $u(t)$ on the interval $[0, T]$

Remarks

- Observability must respect *causality*: only get to look at past measurements
- Each initial condition must generate a *unique* output y
- Start with ignoring noise, disturbances \Rightarrow estimate exact state
- Intuitive way to check observability:

$$\begin{aligned} \dot{x} &= Ax + Bu & y &= Cx \\ y &= Cx & \dot{y} &= C\dot{x} = CAx + CBu \\ & & \ddot{y} &= CA^2x + CABu + CB\dot{u} \\ & & & \vdots \\ & & & CA^{n-1}x \end{aligned} \quad W_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Thm A linear system is observable iff the observability matrix W_o is full rank

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Proof of Observability Rank Condition, 1/2

Thm A linear system is observable if and only if the observability matrix W_o is full rank.

Proof (sufficiency) Write the output in terms of the convolution integral

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t).$$

Since we know $u(t)$, we can subtract off its contribution and write

$$\tilde{y}(t) = Ce^{At}x(0) \quad \tilde{y}(t) = y(t) - \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau - Du(t)$$

Now differentiate the (new) output and evaluate at $t = 0$

$$\tilde{y}(0) = Cx(0)$$

$$\dot{\tilde{y}}(0) = CAx(0)$$

⋮

$$\tilde{y}^{(n)}(0) = CA^{n-1}x(0)$$

Finally, invert to solve for $x(0)$. To find $x(T)$, use $x(T) = e^{AT}x(0)$.

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Proof of Observability Rank Condition, 2/2

Thm A linear system is observable if and only if the observability matrix W_o is full rank.

Proof (necessity) Again, we start with the convolution integral

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t).$$

Subtracting off the input as before and expanding the exponential, we have

$$\tilde{y}(t) = Ce^{At}x(0) = C(I + At + \frac{1}{2}A^2t^2 + \dots + \frac{1}{k!}A^k t^k + \dots)x(0)$$

By the Cayley-Hamilton theorem, we can write A^n in terms of lower powers of A ($A = c_0I + c_1A + c_2A^2 + \dots + c_{n-1}A^{n-1}$) and so we can write

$$\tilde{y}(t) = (\alpha_0(t)C + \alpha_1(t)CA + \dots + \alpha_{n-1}(t)CA^{n-1})x(0)$$

If W_o is not full rank, then can choose $x(0) \neq 0$ such that $\tilde{y}(t) = 0$
 \Rightarrow not observable (since $x(0) = 0$ would produce the same output).

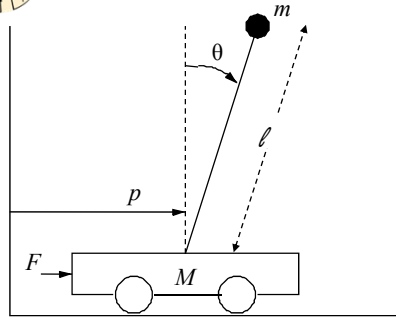
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Example #1: Linearized pendulum on a cart



Question: can we determine the state of the system $(p, \dot{p}, \theta, \dot{\theta})$ from measuring position p ? How about from θ ?

Approach: look at the linearization around the upright position (good approximation to the full dynamics if θ remains small)

M_t = total cart mass
 J_t = total cart inertia

$$\frac{d}{dt} \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2 \ell^2 g}{M_t J_t - m^2 \ell^2} & \frac{-c J_t}{M_t J_t - m^2 \ell^2} & \frac{-\gamma J_t \ell m}{M_t J_t - m^2 \ell^2} \\ 0 & \frac{M_t m g \ell}{M_t J_t - m^2 \ell^2} & \frac{-c \ell m}{M_t J_t - m^2 \ell^2} & \frac{-\gamma M_t}{M_t J_t - m^2 \ell^2} \end{bmatrix} \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{J_t}{M_t J_t - m^2 \ell^2} \\ \frac{\ell m}{M_t J_t - m^2 \ell^2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$$

c = cart damping
 γ = pendulum damping

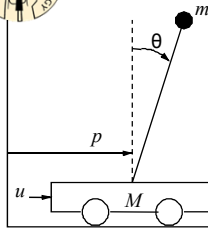
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Example #1, con't: Linearized pendulum on a cart



$$\frac{d}{dt} \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2 \ell^2 g}{\mu} & \frac{-c J_t}{\mu} & \frac{-\gamma J_t \ell m}{\mu} \\ 0 & \frac{M_t m g \ell}{\mu} & \frac{-c \ell m}{\mu} & \frac{-\gamma M_t}{\mu} \end{bmatrix} \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{J_t}{\mu} \\ \frac{\ell m}{\mu} \end{bmatrix} u$$

$\mu = M_t J_t - m^2 \ell^2$

• Simplify by setting $c, \gamma = 0$

Observability matrix (for $C = [1 \ 0 \ 0 \ 0]$)

$$W_o = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{m^2 \ell^2 g}{\mu} & 0 & 0 \\ 0 & 0 & 0 & \frac{m^2 \ell^2 g}{\mu} \end{bmatrix} \begin{matrix} C \\ CA \\ CA^2 \\ CA^3 \end{matrix}$$

• Full rank \Rightarrow observable

• How about with $C = [0 \ 1 \ 0 \ 0]$?

$$W_o = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{M_t m g \ell}{\mu} & 0 & 0 \\ 0 & 0 & 0 & \frac{M_t m g \ell}{\mu} \end{bmatrix}$$

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State Estimation: Full Order Observer

Given that a system is observable, how do we actually estimate the state?

- Key insight: if current estimate is correct, follow the dynamics of the system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad \left\{ \begin{array}{l} \dot{\hat{x}} = \underbrace{A\hat{x} + Bu}_{\text{prediction (copy of dynamics)}} + L(y - C\hat{x}) \leftarrow \text{correction (based on output error)} \end{array} \right.$$

Luenberger Observer

- Modify the dynamics to correct for error based on a linear feedback term
- $L = \mathbf{observer\ gain\ matrix}$; determines how to adjust the state due to error
- Look at the error dynamics for $\tilde{x} = x - \hat{x}$ to determine how to choose L :

$$\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = Ax + Bu - (A\hat{x} + Bu + LC(x - \hat{x})) = (A - LC)\tilde{x}$$

Thm: If pair (A, C) is observable (associated W_o is full rank), we can place the eigenvalues of $A - LC$ arbitrarily through appropriate choice of L .

Proof: Note that the transpose of $A - LC$ is $A^T - C^T L^T$ and in this form, this is the same as the eigenvalue placement problem for state space controllers.

Remark: In MATLAB, use `L = place(A', C', eigs)` to determine L

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Comparison with Reachability & State Fdbk

- 1) The pair (A, C) is observable $\Leftrightarrow (A^T, C^T)$ is reachable
 - True for all reachability tests; e.g. Gramian or PBH test
 - Proofs are similar
- 2) Dynamics with state feedback K are $(A - BK)$
Observer dynamics with gain L are $(A - LC)$
- 3) The poles of $(A - LC)$ can be chosen arbitrarily \Leftrightarrow The poles of $(A^T - C^T L^T)$ can be chosen arbitrarily (see p. 209-211)
 - Use `place(A^T, C^T, lambda)`
 - Convergence depends on $\text{Re}(\lambda)$
- 4) If (A, B) reachable, can transform to reachable canonical form.
If (A, C) observable, can transform to observable canonical form (p. 205).
(Neither observability nor reachability depend on a change of variables)

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Simple Example

Double-integrator: $\ddot{z} = u$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u$$

$$y = x_1$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Measure position, estimate full state
(position and velocity)

Guess: $\hat{x}_1 = y, \hat{x}_2 = \dot{y}$

$$W_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Design observer:

$$\dot{\hat{x}} = A\hat{x} - L(Cx - C\hat{x}) + Bu$$

$$L = \text{place}(A', C', [-1; -1]) = \begin{bmatrix} 2 & 1 \end{bmatrix};$$

$$A - LC = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\dot{\hat{x}}_1 = -2\hat{x}_1 + \hat{x}_2 + 2y$$

$$\dot{\hat{x}}_2 = -\hat{x}_1 + y + u$$

```

Wo=obsv(A,C);
L=place(A',C',[-1;-1.00001]);
sys=ss(A,B,C,0);
[u,t]=gensig('sin',5,20,dt);
[y,t,x]=lsim(sys,u,t);
y=y+0.1*randn(size(y));
% simple observer:
xh1=y;
xh2=diff([0;y])/dt;
% Luenberger observer:
Ao=A-L*C;
sys0=ss(Ao,[L B],eye(2),zeros(2,2));
xh=lsim(sys0,[y u],t);

```

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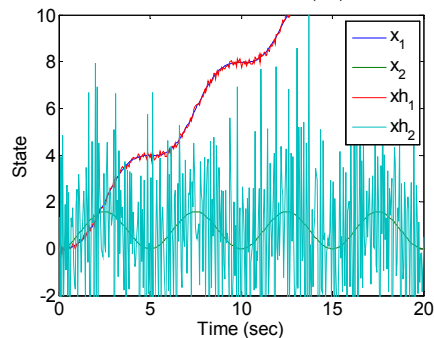
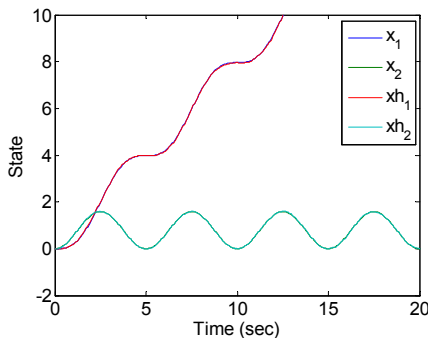
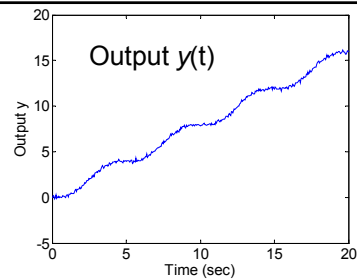
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Guess which observer is which?

a) $\hat{x}_1 = y, \hat{x}_2 = \dot{y}$

b) $\dot{\hat{x}} = A\hat{x} - L(Cx - C\hat{x}) + Bu$



$$\dot{\hat{x}} = A\hat{x} - L(Cx - C\hat{x}) + Bu$$

$$\hat{x}_1 = y, \hat{x}_2 = \dot{y}$$

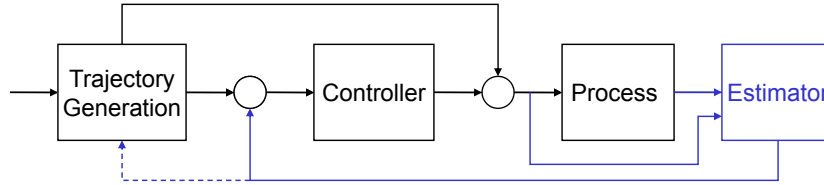
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Separation Principle



What happens when we apply state space controller using *estimate* of x ?

- We assumed we measured x directly in analyzing controller; extra dynamics in the estimator could cause closed loop to go unstable

Thm: If K is a stabilizing compensator for (A, B) and L gives a stable estimator for (A, C) , then the control law $u = -K\hat{x}$ is stable (see p. 212-213)

- This is an example of a *separation principle*: design the controller and estimator separately, then combine them and everything is OK
- Be careful with signs on gains

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Proof of Separation Theorem

Proof. Write down the dynamics for the complete system (assuming WLOG that $x_d, u_d = 0$):

$$\begin{aligned} \dot{x} &= Ax + Bu & \dot{\hat{x}} &= A\hat{x} + Bu + L(y - C\hat{x}) \\ y &= Cx & u &= -K\hat{x} + u_{\text{ref}} \end{aligned}$$

Rewrite in terms of the error dynamics $\tilde{x} = x - \hat{x}$ and combined state x, \tilde{x} :

$$\dot{\tilde{x}} = (A - LC)\tilde{x} \quad \frac{d}{dt} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} u_{\text{ref}} \\ 0 \end{bmatrix}$$

Since the dynamics matrix is block diagonal, we find that the characteristic polynomial of the closed loop system is

$$\det(sI - A + BK) \det(sI - A + LC).$$

This polynomial is a product of two terms, where the first is the characteristic polynomial of the closed loop system obtained with state feedback and the other is the characteristic polynomial of the observer error.

Since each was designed to be stable \Rightarrow the entire system is stable

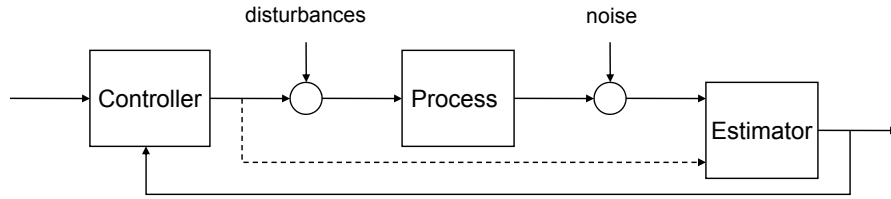
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Summary: Observers and State Estimation



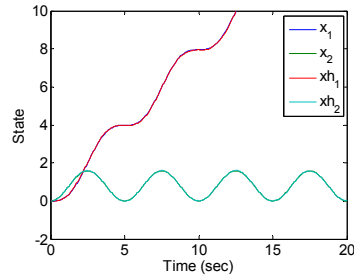
Observability

- Derived conditions for when we could determine state from inputs & outputs: check rank of observability matrix

State Estimators

- Construct state estimate based on prediction and correction (no noise yet)

CDS110b: add noise to the problem formulation \Rightarrow Kalman filter



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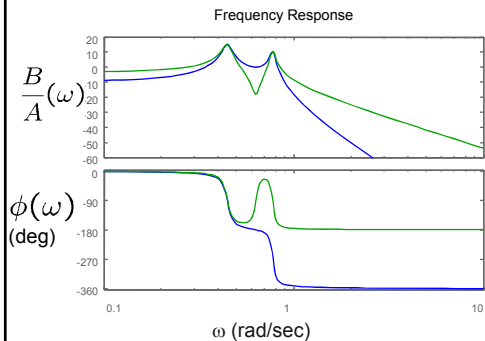
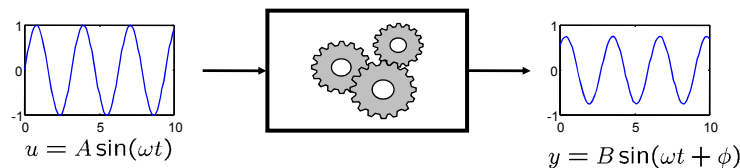
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Frequency Domain Modeling (Chapter 8)

Defn. The *frequency response* of a linear system is the relationship between the gain and phase of a sinusoidal input and the corresponding steady state (sinusoidal) output.



Bode plot (1940; Henrik Bode)

- Plot gain and phase vs input frequency
- Gain is plotting using log-log plot
- Phase is plotting with log-linear plot
- Can read off the system response to a sinusoid – in the lab or in simulations
- Linearity \Rightarrow can construct response to any input (via Fourier decomposition)
- Key idea: do all computations in terms of gain and phase (frequency domain)

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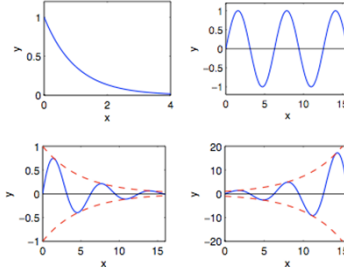
Transmission of Exponential Signals

Exponential signal: $e^{st} = e^{(\sigma+i\omega)t} = e^{\sigma t} e^{i\omega t} = e^{\sigma t} (\cos \omega t + i \sin \omega t)$

- Construct constant inputs + sines/cosines by linear combinations
 - Constant: $u(t) = c = ce^{0t}$
 - Sinusoid: $u(t) = A \sin(\omega t) = \frac{A}{2i} (e^{i\omega t} - e^{-i\omega t})$
- Exponential response can be computed via the convolution equation

$$\begin{aligned}
 x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)} B e^{s\tau} d\tau \\
 &= e^{At}x(0) + e^{At}(sI - A)^{-1} e^{(sI-A)\tau} \Big|_{\tau=0}^t B \\
 &= e^{At}x(0) + e^{At}(sI - A)^{-1} (e^{(sI-A)t} - I) B \\
 &= e^{At} (x(0) - (sI - A)^{-1} B) + (sI - A)^{-1} B e^{st}
 \end{aligned}$$

$$\begin{aligned}
 y(t) &= Cx(t) + Du(t) \\
 &= C e^{At} (x(0) - (sI - A)^{-1} B) + (C(sI - A)^{-1} B + D) e^{st}
 \end{aligned}$$



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Transfer Function and Frequency Response

Exponential response of a linear state space system

$$y = \underbrace{C e^{At} (x(0) - (sI - A)^{-1} B)}_{\text{transient}} + \underbrace{(C(sI - A)^{-1} B + D) e^{st}}_{\text{steady state}}$$

Transfer function

- Steady state response is proportional to exponential input => look at input/output ratio
- $G(s) = C(sI - A)^{-1} B + D$ is the *transfer function* between input and output

Frequency response

$$\begin{aligned}
 u(t) &= A \sin \omega t = \frac{A}{2i} (e^{i\omega t} - e^{-i\omega t}) \\
 y_{ss}(t) &= \frac{A}{2i} (G(i\omega) e^{i\omega t} - G(-i\omega) e^{-i\omega t}) \\
 &= A \cdot |G(i\omega)| \sin(\omega t + \arg G(i\omega))
 \end{aligned}$$

gain
phase

Common transfer functions

$\dot{y} = u$	$\frac{1}{s}$
$y = \dot{u}$	s
$\dot{y} + ay = u$	$\frac{1}{s+a}$
$\ddot{y} = u$	$\frac{1}{s^2}$
$\ddot{y} + 2\xi\omega_n\dot{y} + \omega_n^2 y = u$	$\frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2}$
$y = k_p u + k_d \dot{u} + k_i \int u$	$k_p + k_d s + \frac{k_i}{s}$
$y(t) = u(t - \tau)$	$e^{-\tau s}$

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