Lecture 4.2

Reachability & State Fdbk

Topics

1. Reachability
   - Cayley-Hamilton
   - Gramian & PBH Tests
2. Canonical Form & pole placement
3. Reference input & integral control

Reachability

A system $\dot{x} = f(x, u)$ is reachable if $\forall x_0, x_F$ and any $T > 0$ $\exists u^*(t)$ such that the sol'n of the dynamics with $x(0) = x_0$ & $u = u^*(t)$ gives $x(T) = x_F$.

For linear systems

The system $\dot{x} = Ax + Bu$ with $x \in \mathbb{R}^n$ is reachable if

$$P = \begin{bmatrix} B & AB & A^2B \cdots & A^{n-1}B \end{bmatrix}$$

is full rank

Remarks:

1. Depends only on A, B, not C
2. Can define the reachable subspace as the image of P
3. Doesn't depend on x_0 or T

Find $u(t)$ $\ni$ $x(T) = \int_0^T e^{A(T-\tau)} u(\tau)\, d\tau$

DGM
10/22/08

Cayley - Hamilton Thm :

For any $A \in \mathbb{R}^{n \times n}$, with characteristic polynomial

$$\det(sI - A) = s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n$$

Then i) $A^n + a_1 A^{n-1} + \dots + a_{n-1} A + a_n I = 0$

ii) A^k, $k \geq n$, is a linear combination of $A^0 \dots A^{n-1}$

$$\left(A^k = \sum_{j=0}^{n-1} \alpha_j A^j \right)$$

PF (sketch) : For any eigenvalue λ of A

Then $\lambda^n + a_1 \lambda^{n-1} + \dots + a_n = 0$

$\Rightarrow$ diagonalize A and (i) Follows

(ii) Follows immediately From (i)

Remarks:

1) True for general (non-diagonalizable) A Too

2) Follows That

$$\mathrm{rank}\left(\left[B \;\; AB \;\; A^2B \; \cdots \; A^{n-1}B \;\; A^nB \; \cdots \right] \right)$$

$$= \mathrm{rank}\left(\left[B \;\; AB \;\; A^2B \; \cdots \; A^{n-1}B \right] \right)$$

Write $e^{At} = I + At + \frac{1}{2} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \dots$

$$= \phi_0(t) I + \phi_1(t) A + \dots + \phi_{n-1}(t) A^{n-1}$$

For some scalar Functions $\phi(t)$

So $x(T) = \int_0^T e^{A(t-\tau)} B u(\tau) d\tau = \int_0^T \phi_0(t-\tau) B u(\tau) d\tau + \dots + \int_0^T \phi_{n-1}(t-\tau) A^{n-1} B u(\tau) d\tau$

$$= \left[B \;\; AB \; \cdots \; A^{n-1}B \right] \begin{bmatrix} \int \phi_0 u \\ \vdots \\ \int \phi_n u \end{bmatrix}$$

$\Rightarrow$ Proves necessity

(Full rank $\Leftarrow$ controllable)

PBH remarks

1. Again, one direction of proof is easy to show:

Suppose rank $\begin{bmatrix} A-\lambda I & B \end{bmatrix} < n$

Then $\exists\, x \neq 0, \quad x^T \begin{bmatrix} A-\lambda I & B \end{bmatrix} = 0$

$$\Rightarrow \underbrace{x^T A = \lambda x^T} \quad \& \quad x^* B = 0$$

$$\Rightarrow x^T A^2 = \lambda^2 x^T$$

$$\vdots$$

$$x^T A^{n-1} = \lambda^{n-1} x^T$$

Hence $\quad x^T \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = 0$

2. PBH identifies uncontrollable modes

Reachability Canonical Form
(and state feedback via pole placement)

$$\dot{x} = Ax + Bu \quad , \quad x \in \mathbb{R}^n$$
$$y = Cx + Du$$

Change of variables $z = Tx$, $T \in \mathbb{R}^{n \times n}$, invertible

Then
$$\dot{z} = (TAT^{-1}) z + TB u$$
$$y = CT^{-1} z + Du$$

Note : if $A_z = TAT^{-1}$, $B_z = TB$

Then $\begin{bmatrix} B_z & A_z B_z & \cdots & A_z^{n-1} B_z \end{bmatrix} = T \begin{bmatrix} B & AB & \cdots & A^{n-1} B \end{bmatrix}$

$\Rightarrow$ Reachability does not depend on change of variables.

__Thm:__ If (A,B) reachable, then $\exists T$, $z = Tx$, such that

$$\dot{z} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & \\ 0 & 1 & \cdots & \\ \vdots & & & \\ 0 & \cdots & 1 & 0 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u \qquad (\text{for } u \in \mathbb{R})$$

$$\dot{z}_n = z_{n-1}$$
$$\dot{z}_{n-1} = z_{n-2}$$
$$\vdots$$
$$\dot{z}_1 = -a_1 z_1 - a_2 z_2 - \cdots$$

or $\frac{d^n}{dt^n} z + a_1 \frac{d^{n-1}}{dt^{n-1}} z + \cdots$

$\det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_n$

In these co-ordinates,

$$\tilde{W}_r = \begin{bmatrix} 1 & -a_1 & a_1^2 - a_2 & \\ 0 & 1 & -a_1 & \\ \vdots & & 1 & \\ & & & \ddots \end{bmatrix}$$

Note $\tilde{W}_r = T W_r$ (from above)

So $T = \tilde{W}_r W_r^{-1}$ (if $\det(W_r) \neq 0$)

Pole Placement

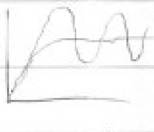
Thm: (A,B) reachable $\Leftrightarrow$ closed-loop eigenvalues can be placed arbitrarily through state feedback $u = -Kx$

Pf: In reachable canonical form, choose

$$\tilde{K} = [p_1 - a_1 \quad p_2 - a_2 \quad \cdots \quad p_n - a_n]$$

where $\lambda(s) = s^n + a_1 s^{n-1} + \dots + a_n$ is the characteristic poly. of A and $p(s) = s^n + p_1 s^{n-1} + \dots + p_n$ is the desired ch. p.

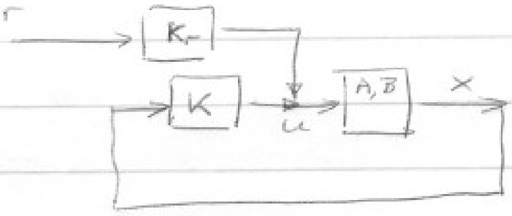
Then $K = \tilde{K} T = \tilde{K} \tilde{W}_r W_r^{-1}$. This is Ackermann's Formula



Remarks

1. Response-Time depends on eigenvalue: $\lambda = \sigma \pm j\omega$ leads to $e^{-\sigma t}(a \sin \omega t + b \cos \omega t)$ (see Table 6.1)
2. Response Time dominated by "slowest" eigenvalue:
   compare step resp. of $\dot{x} = -x + u$, $y = x$
   with " " " $\dot{x}_1 = -x_1 + u$
   $\dot{x}_2 = +10x_1 - 10x_2$, $y = x_2$
   $\left(A = \begin{bmatrix} -1 & 0 \\ -10 & -10 \end{bmatrix}, \ \mathrm{eig}(A) = -1, -10 \text{ and } CA^{-1}B = 1 \right)$
3. Faster response requires more control $\Rightarrow$ choose pole as fast as needed, no faster
4. CDS 110b: "linear quadratic regulator", Find K that minimizes $J = \int_0^\infty x^T Q x + u^T R u \, dt$
5. $u = -Kx$ changes dynamics to $\dot{x} = (A - BK)x$ but doesn't track a reference input

Reference Tracking



Choose $u = -Kx + k_r r$

So that output

y tracks reference r

$$\dot{x} = Ax + B(-Kx + k_r r)$$
$$y = Cx$$

In steady-state (assuming stable), $\dot{x} = 0$

$$\Rightarrow y = -C(A - BK)^{-1} k_r r$$

- This requires careful calibration of the gain (not robust)

Integral action

Define $\dot{z} = y - r$, control $u = -Kx - k_i z + k_r r$

Then in steady state (if stable), $\dot{z} = 0 \Rightarrow y = r$

Augment state:

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$

& design $u = -\begin{bmatrix} K & k_i \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix}$

to stabilize augmented system.

Remarks

1. With $k_r = 0$, still get perfect tracking
2. Also get perfect tracking with steady disturbance (not in general true without integral gain)