



CDS 101: Lecture 4-1

Reachability and State Space Feedback

Douglas G. MacMynowski

Goals:

- Define reachability of a control system
- Give tests for reachability of linear systems and apply to examples
- Describe the design of state feedback controllers for linear systems

Reading:

- Åström and Murray, *Feedback Systems*, Ch 6

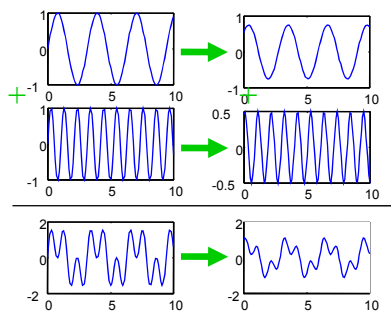
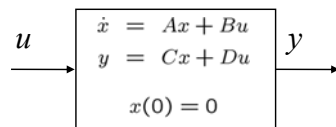
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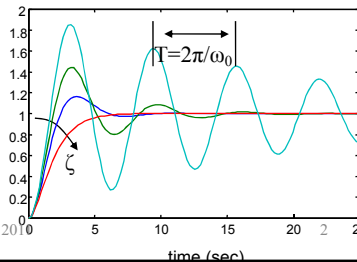
Review from last week



Properties of Linear Systems:

- Linearity with respect to inputs and initial conditions
- Stability characterized by eigenvalues
- Response described by convolution integral
- Many applications and tools
- Characterizes a nonlinear system around an equilibrium point
- *No system is linear for all input amplitudes, but linearization is a good enough model for many systems*

$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

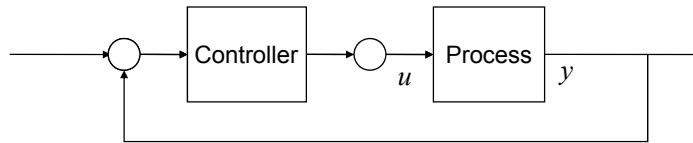


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Control Overview



- Design controller so that:
 - i) System is stable
 - ii) Performance:
 - Keep close to equilibrium despite disturbances (disturbance rejection)
 - Move system to desired state (track reference)
 - iii) Robust to modeling errors
- Design framework:
 - Use time-domain (state-space) model (week 4-5, and cds110b)
 - Use frequency-domain information (weeks 5-10)



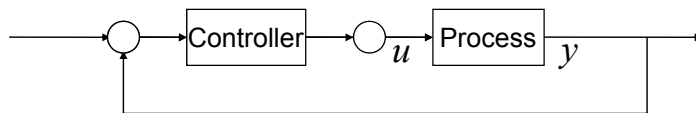
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First Questions...



- 1) Can the input u affect the dynamics?

$$\begin{aligned} \dot{x}_1 &= x_1 + u \\ \text{e.g. } \dot{x}_2 &= x_2 \quad \Rightarrow \text{Can't change } x_2 \end{aligned}$$

Equivalent to asking whether there is a u that allows us to reach any point in the state-space

- ⇒ Reachability (today), depends on A, B
- ⇒ Related to the design of state feedback

- 2) Does the measurement y contain enough information about the system?

$$\begin{aligned} \dot{x}_1 &= x_1 \quad y = x_1 \\ \text{e.g. } \dot{x}_2 &= x_2 \quad \Rightarrow \text{Can't measure } x_2 \end{aligned}$$

- ⇒ Observability (next week), depends on A, C
- ⇒ Related to the design of observers to estimate state from measurement

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Control Design Concepts

- System description: often single input, single output (MIMO also OK)

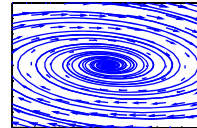
$$\dot{x} = f(x, u) \quad x \in \mathbb{R}^n, x(0) \text{ given}$$

$$y = h(x, u) \quad u \in \mathbb{R}, y \in \mathbb{R}$$

- Stability: stabilize the system around an equilibrium point

- Given equilibrium point $x_e \in \mathbb{R}^n$, find control “law”

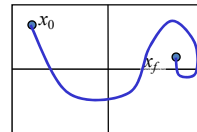
$$u = \alpha(x) \text{ such that } \lim_{t \rightarrow \infty} x(t) = x_e \quad \forall x(0) \in \mathbb{R}^n$$



- Reachability: steer the system between two points

- Given $x_0, x_f \in \mathbb{R}^n$, find an input $u(t)$ such that

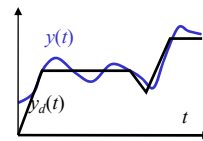
$$\dot{x} = f(x, u(t)) \text{ takes } x(t_0) = x_0 \rightarrow x(T) = x_f$$



- Tracking: track a given output trajectory

- Given $y_d(t)$, find $u = \alpha(x, t)$ such that

$$\lim_{t \rightarrow \infty} (y(t) - y_d(t)) = 0, \quad \forall x(0) \in \mathbb{R}^n$$



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Reachability of Input/Output Systems

$$\dot{x} = f(x, u) \quad x \in \mathbb{R}^n, x(0) \text{ given}$$

$$y = h(x, u) \quad u \in \mathbb{R}, y \in \mathbb{R}$$

- **Def'n:** An input/output system is *reachable* if for any $x_0, x_f \in \mathbb{R}^n$ and any time $T > 0$ there exists an input $u_{[0, T]} \in \mathbb{R}^n$ such that the solution of the dynamics starting from $x(0)=x_0$ and applying input $u(t)$ gives $x(T)=x_f$.

- Remarks

- In the definition, x_0 and x_f do not have to be equilibrium points

⇒ we don't necessarily stay at x_f after time T .

- Reachability is defined in terms of states ⇒ doesn't depend on output

- For *linear systems*, can characterize reachability by looking at the general solution:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$x(T) = e^{AT}x_0 + \int_{\tau=0}^T e^{A(T-\tau)}Bu(\tau)d\tau$$

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Tests for Reachability

$$\begin{aligned} \dot{x} &= Ax + Bu & x \in \mathbb{R}^n, x(0) \text{ given} \\ y &= Cx + Du & u \in \mathbb{R}, y \in \mathbb{R} \end{aligned} \quad x(T) = e^{AT}x_0 + \int_{\tau=0}^T e^{A(T-\tau)}Bu(\tau)d\tau$$

- **Thm:** A linear system is reachable if and only if the $n \times n$ reachability matrix

$$\begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$
 is full rank.

- **Remarks**

- Very simple test to apply. In MATLAB, use `ctrb(A,B)` and check rank w/ `det()`
- If this test is satisfied, we say “the pair (A,B) is reachable”
- Some insight into the proof can be seen by expanding the matrix exponential

$$\begin{aligned} e^{A(T-\tau)}B &= \left(I + A(T-\tau) + \frac{1}{2}A^2(T-\tau)^2 + \dots + \frac{1}{(n-1)!}A^{n-1}(T-\tau)^{n-1} + \dots \right) B \\ &= B + AB(T-\tau) + \frac{1}{2}A^2B(T-\tau)^2 + \dots + \frac{1}{(n-1)!}A^{n-1}B(T-\tau)^{n-1} + \dots \end{aligned}$$

- Test does not give a measure of how much control effort is required
- Other tests for reachability also exist

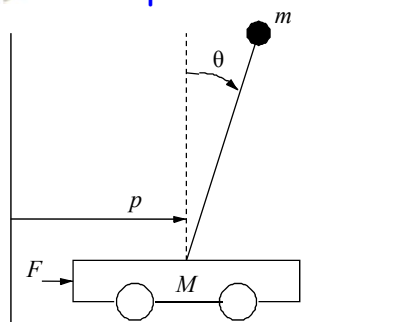
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Example #1: Linearized pendulum on a cart



- Question: can we locally control the position of the cart by proper choice of input?
- Approach: look at the linearization around the upright position (good approximation to the full dynamics if θ remains small)


$$\frac{d}{dt} \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2 l^2 g}{M_t J_t - m^2 l^2} & \frac{-c J_t}{M_t J_t - m^2 l^2} & \frac{-\gamma J_t l m}{M_t J_t - m^2 l^2} \\ 0 & \frac{M_t m g l}{M_t J_t - m^2 l^2} & \frac{-c l m}{M_t J_t - m^2 l^2} & \frac{-\gamma M_t}{M_t J_t - m^2 l^2} \end{bmatrix} \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{J_t}{M_t J_t - m^2 l^2} \\ \frac{l m}{M_t J_t - m^2 l^2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x,$$

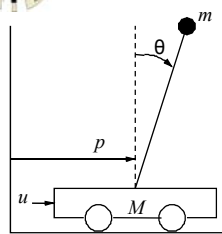
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Example #1, con't: Linearized pendulum on a cart



$$\frac{d}{dt} \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2 l^2 g}{\mu} & \frac{-c J_t}{\mu} & \frac{-\gamma J_t l m}{\mu} \\ 0 & \frac{M_t m g l}{\mu} & \frac{-c l m}{\mu} & \frac{-\gamma M_t}{\mu} \end{bmatrix} \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{J_t}{\mu} \\ \frac{l m}{\mu} \end{bmatrix} u$$

$\mu = M_t J_t - m^2 l^2$

• Simplify by setting $c, \gamma = 0$

• Reachability matrix


$$W_r = \begin{bmatrix} 0 & \frac{J_t}{\mu} & 0 & 0 \\ 0 & \frac{l m}{\mu} & 0 & 0 \\ \frac{J_t}{\mu} & 0 & \frac{g l^3 m^3}{\mu^2} & 0 \\ \frac{l m}{\mu} & 0 & \frac{g l^2 m^2 (m+M)}{\mu^2} & 0 \end{bmatrix} \begin{bmatrix} B \\ AB \\ A^2 B \\ A^3 B \end{bmatrix}$$

- Full rank as long as constants are such that columns 1 and 3 are not multiples of each other
- Reachable as long as $\det(W_r) = \frac{g^2 l^4 m^4}{\mu^4} \neq 0$
- Can "steer" linearization between points by proper choice of input

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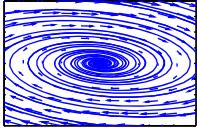
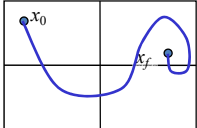
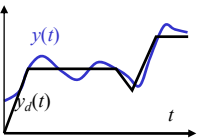
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- Reachability: steer the system between two points
 - Given $x_0, x_f \in \mathbb{R}^n$, find an input $u(t)$ such that $\dot{x} = f(x, u(t))$ takes $x(t_0) = x_0 \rightarrow x(T) = x_f$
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State space controller design for linear systems

$$\begin{aligned} \dot{x} &= Ax + Bu & x \in \mathbb{R}^n, x(0) \text{ given} \\ y &= Cx + Du & u \in \mathbb{R}, y \in \mathbb{R} \end{aligned} \quad x(T) = e^{AT}x_0 + \int_{\tau=0}^T e^{A(T-\tau)}Bu(\tau)d\tau$$

- **Goal:** find a linear control law $u = -Kx$ such that the closed loop system

$$\dot{x} = Ax - BKx = (A - BK)x$$

is stable at $x_e=0$.

- **Remarks**
 - Stability based on eigenvalues \Rightarrow use K to make eigenvalues of $(A - BK)$ stable
 - Can also link eigenvalues to *performance* (e.g., initial condition response)
 - Question: when can we place the eigenvalues any place that we want?
- **Theorem:** The eigenvalues of $(A - BK)$ can be set to arbitrary values if and only if the pair (A, B) is reachable.
- **MATLAB:** $K = \text{place}(A, B, \text{eigs})$

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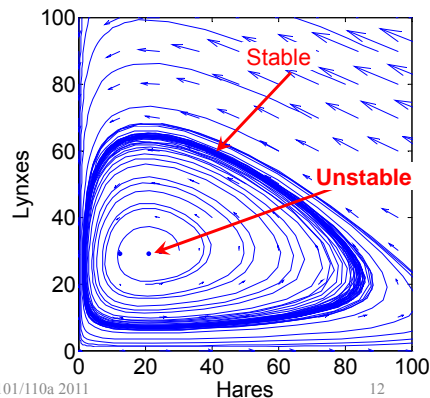
Example #2: Predator prey

$$\begin{aligned} \frac{dH}{dt} &= rH \left(1 - \frac{H}{k_c}\right) - \frac{aHL}{c + H}, & H \geq 0 \\ \frac{dL}{dt} &= b \frac{aHL}{c + H} - dL, & L \geq 0 \end{aligned}$$

- Controlled dynamics: modulate food supply

$$\begin{aligned} \frac{dH}{dt} &= (r + u)H \left(1 - \frac{H}{k_c}\right) - \frac{aHL}{c + H} \\ \frac{dL}{dt} &= b \frac{aHL}{c + H} - dL, \end{aligned}$$

- Q1: can we move from some initial population of lynxes and hares to a specified one in time T by modulation of the food supply?
- Q2: can we stabilize the population around the desired equilibrium point
- Approach: try to answer this question locally, around the natural equilibrium point



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Example #2: Problem setup

- Equilibrium point calculation

$$\frac{dH}{dt} = (r + u)H \left(1 - \frac{H}{k_c}\right) - \frac{aHL}{c + H}$$

$$\frac{dL}{dt} = b \frac{aHL}{c + H} - dL,$$

$$- x_e = (20.5, 29.5), u_e = 0$$

- Linearization

- Compute linearization around equil.

point, x_e :

$$A = \left. \frac{\partial f}{\partial x} \right|_{x_e, u_e} \quad B = \left. \frac{\partial f}{\partial u} \right|_{x_e, u_e}$$

```
% Compute the equil point
% predprey.m contains dynamics
f = inline('predprey(0,x)');
xeq = fsolve(f, [20, 30]);
```

```
% Compute linearization
A = [...];
B = [H0*(1 - H0/K); 0];
p = [-1;-2];
K = place(A,B,p)
```

$$\frac{d}{dt} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} r - \frac{2H_0 r}{k} - \frac{aL_0}{c+H_0} + \frac{aL_0 H_0}{(c+H_0)^2} & -\frac{aH_0}{c+H_0} \\ baL_0 \left(\frac{1}{c+H_0} - \frac{H_0}{(c+H_0)^2} \right) & ba \frac{H_0}{c+H_0} - d \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} H_0 \left(1 - \frac{H_0}{k}\right) \\ 0 \end{bmatrix} v$$

- Reachable? YES, if $ba \neq 0$ (check $[B \ AB]$) \Rightarrow can locally steer to any point

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Example #2: Stabilization via eigenvalue assignment

$$\frac{d}{dt} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} r - \frac{2H_0 r}{k} - \frac{aL_0}{c+H_0} + \frac{aL_0 H_0}{(c+H_0)^2} & -\frac{aH_0}{c+H_0} \\ baL_0 \left(\frac{1}{c+H_0} - \frac{H_0}{(c+H_0)^2} \right) & ba \frac{H_0}{c+H_0} - d \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} H_0 \left(1 - \frac{H_0}{k}\right) \\ 0 \end{bmatrix} v$$

- Control design:

$$v = -Kz + k_r r$$

$$u = u_e - K(x - x_e) + k_r(r - y_e)$$

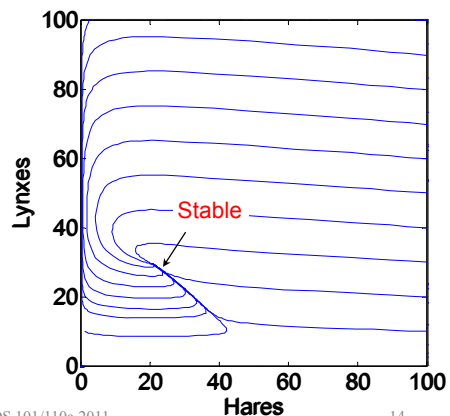
- Place poles at stable values

- Choose $\lambda = -1, -2$
- $K = \text{place}(A, B, [-1; -2]);$

- Modify NL dynamics to include control

$$\frac{dH}{dt} = (r + u)H \left(1 - \frac{H}{k_c}\right) - \frac{aHL}{c + H}$$


$$\frac{dL}{dt} = b \frac{aHL}{c + H} - dL,$$



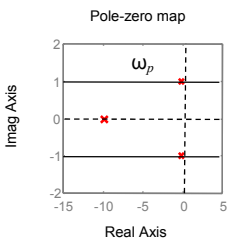
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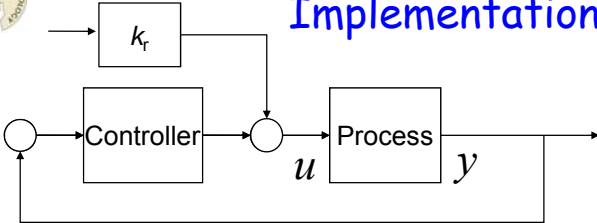
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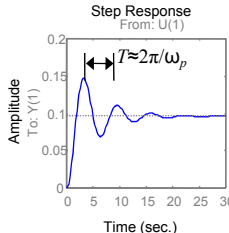
Implementation






Remarks:

- In practice, don't always have access to full state
→ estimate state from measurement y (next week)
- What to pick for eigenvalues?
 - For each eigenvalue $\lambda_i = \sigma_i + j\omega_i$, get contribution of the form $y_i(t) = e^{\sigma_i t} (a \sin(\omega_i t) + b \cos(\omega_i t))$
 - Faster response will require more control effort
 - Optimal control: LQR (in text, CDS 110b)
- How to obtain desired tracking response so that $y_{ss} = r$ for some reference r ?
 - Choose $u = -Kx + k_r r$
 - Steady state (if $D=0$): $y = Cx = C(A - BK)^{-1} B k_r r \Rightarrow k_r = \text{inv}[C(A - BK)^{-1} B]$



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Reachable Canonical Form

- If the system is reachable, then there exists a transformation $z = Tx$ such that:

$$\dot{z} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & 1 & 0 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$
- (Check $[B \ AB \ A^2B \ \dots \ A^{n-1}B]$... is this system reachable?)
- Characteristic polynomial is: $\lambda(s) = s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n$
- Choose state feedback: $u = -Kz = -[k_1 \ k_2 \ \dots \ k_n] z$
- Then closed-loop system is:

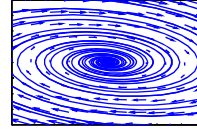
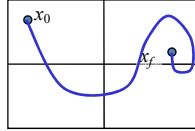
$$\dot{z} = \begin{bmatrix} -a_1 - k_1 & -a_2 - k_2 & -a_3 - k_3 & \cdots & -a_n - k_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & 1 & 0 \end{bmatrix} z$$

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Summary: Reachability and State Space Feedback

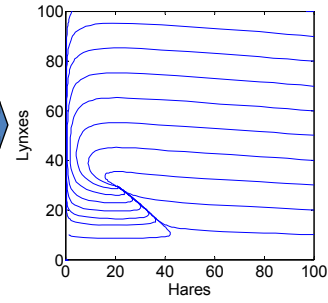
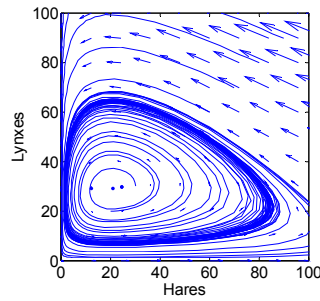
$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$



$$[B \ AB \ A^2B \ \dots \ A^{n-1}B]$$

$$u = -Kx + k_r r$$

- Key concepts
 - Reachability: find u s.t. $x_0 \rightarrow x_f$
 - Reachability rank test for linear systems
 - State feedback to assign eigenvalues



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