# CDS 101/110a: Lecture 3.1
# Linear Systems

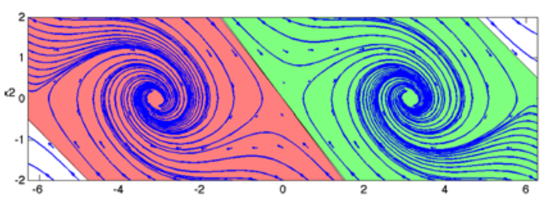
**Douglas G. MacMynowski**

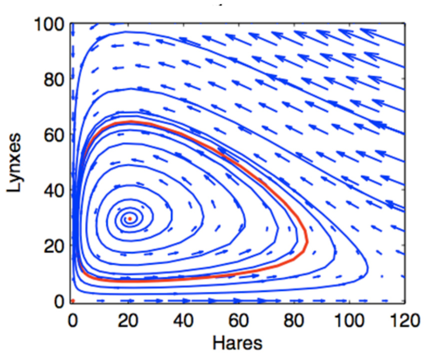
**Goals:**
- Describe linear system models: properties, examples, and tools
  - Convolution equation describing solution in response to an input
  - Frequency response
- Characterize stability and performance of linear systems using eigenvalues
- Compare linearization of a nonlinear systems around an equilibrium point

**Reading:**
- Åström and Murray, Analysis and Design of Feedback Systems, Ch 5

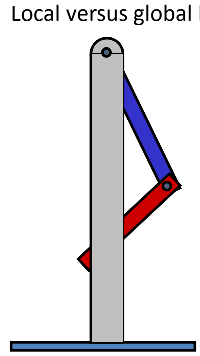


# Summary: Stability and Performance




- Key topics
  - Stability of equilibrium points
  - Eigenvalues determine stability for linear systems
  - Local versus global behavior



# Linearization Around an Equilibrium Point

$$\begin{aligned}\dot{x} &= f(x,u) \\ y &= h(x,u)\end{aligned} \;\Rightarrow\; \begin{aligned}\dot{z} &= Az + Bu \\ y &= Cz + Du\end{aligned}$$

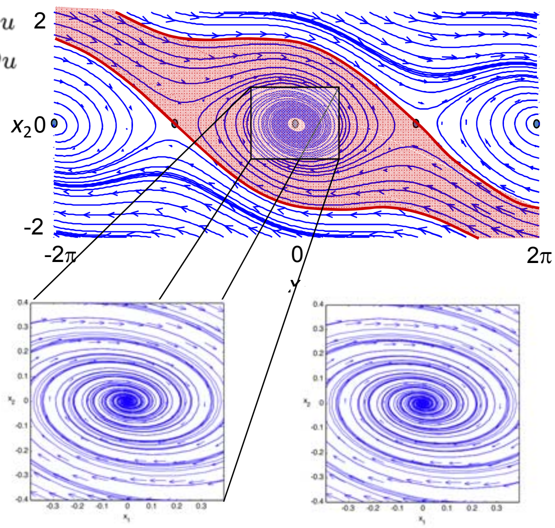
"Linearize" around $x=x_e$

$$f(x_e,u_e)=0 \qquad y_e = h(x_e,u_e)$$

$$z = x - x_e \qquad v = u - u_e \qquad w = y - y_e$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x_e,u_e)} \qquad B = \left.\frac{\partial f}{\partial u}\right|_{(x_e,u_e)}$$

$$C = \left.\frac{\partial h}{\partial x}\right|_{(x_e,u_e)} \qquad D = \left.\frac{\partial h}{\partial u}\right|_{(x_e,u_e)}$$

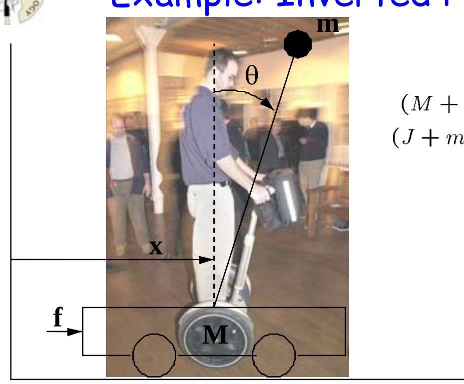



Full nonlinear model    Linear model (honest!)

- Remarks
  - In examples, this is often equivalent to small angle approximations, etc
  - Only works *near* equilibrium point



# Example: Inverted Pendulum on a Cart




$$(M+m)\ddot{x} + ml\cos\theta\ddot{\theta} = -b\dot{x} + ml\sin\theta\dot{\theta}^2 + f$$
$$(J+ml^2)\ddot{\theta} + ml\cos\theta\ddot{x} = -mgl\sin\theta$$

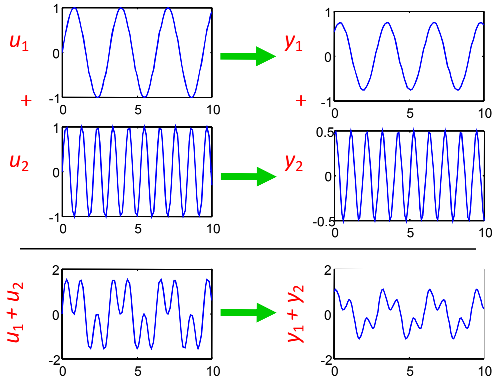
- State: $x,\ \theta,\ \dot{x},\ \dot{\theta}$
- Input: $u = F$
- Output: $y = x$
- Linearize according to previous formula around θ = 0

$$\frac{d}{dt}\begin{bmatrix}x\\ \theta\\ \dot{x}\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & \frac{m^2gl^2}{J(M+m)+Mml^2} & \frac{-(J+ml^2)b}{J(M+m)+Mml^2} & 0\\ 0 & \frac{mgl(M+m)}{J(M+m)+Mml^2} & \frac{-mlb}{J(M+m)+Mml^2} & 0\end{bmatrix}x + \begin{bmatrix}0\\ 0\\ \frac{J+ml^2}{J(M+m)+Mml^2}\\ \frac{ml}{J(M+m)+Mml^2}\end{bmatrix}u$$

$$y = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}x$$

# Linear Systems

$$u \rightarrow \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \\ x(0) &= 0 \end{aligned} \rightarrow y$$



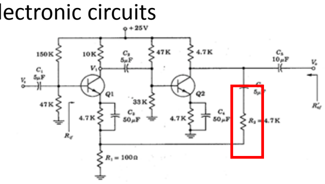

- Input/output linearity at x(0) = 0
  - Linear systems are linear in initial condition and input ⇒ need to use x(0) = 0 to add outputs together
  - For different initial conditions, you need to be more careful
- Linear system ⇒ step response and frequency response scale with input amplitude
  - 2X input ⇒ 2X output
  - Allows us to use ratios and percentages in step or frequency response. *These are independent of input amplitude*
  - Limitation: input saturation ⇒ only holds up to certain input amplitude



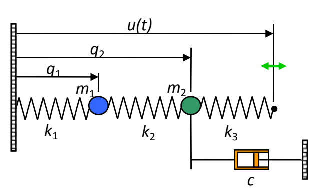
# Why are Linear Systems Important?

**Many important *examples***

- Electronic circuits



  - Especially true after feedback
  - Frequency response is key performance specification
- Many mechanical systems




**Many important *tools***

- Frequency and step response,
  - Traditional tools of control theory
  - Developed in 1930's at Bell Labs
- Classical control design toolbox
  - Nyquist plots, gain/phase margin
  - Loop shaping

(CDS 101/110a)

- Optimal control and estimators
  - Linear quadratic regulators
  - Kalman estimators

(CDS 110b)

- Robust control design
  - $H_\infty$ control design
  - $\mu$ analysis for structured uncertainty

(CDS 212)

# Solutions of Linear Systems: The Matrix Exponential

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned} \longrightarrow y(t) = ???$$

- Scalar linear system, with no input

$$\begin{aligned}\dot{x} &= ax \\ y &= cx\end{aligned} \quad x(0) = x_0 \longrightarrow x(t) = e^{at}x_0 \longrightarrow y(t) = ce^{at}x_0$$

- Matrix version, with no input

$$\begin{aligned}\dot{x} &= Ax \\ y &= Cx\end{aligned} \quad x(0) = x_0 \longrightarrow x(t) = e^{At}x_0 \longrightarrow \boxed{y(t) = Ce^{At}x_0}$$

sys=ss(A,B,C,D);
initial(sys,x0);

- Matrix exponential (→ $e^{At}$)
  - Analog to the scalar case; defined by series expansion:

$$e^M = I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \cdots$$

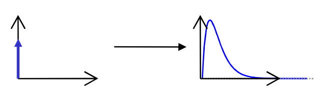
P = expm(M)



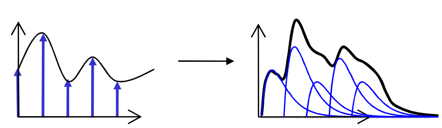
# Response to inputs: Convolution

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned} \longrightarrow y(t) = \underbrace{Ce^{At}x(0)}_{\text{homogeneous}} + ???$$

- *Impulse response*, $h(t) = Ce^{At}B$
  - Response to input "impulse"
  - Equivalent to "Green's function"



- Linearity ⇒ compose response to arbitrary $u(t)$ using *convolution*
  - Decompose input into "sum" of shifted impulse functions
  - Compute impulse response for each
  - "Sum" impulse response to find $y(t)$



- Complete solution: use integral instead of "sum"

$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

- linear with respect to initial condition *and* input
- 2X input ⇒ 2X output when $x(0) = 0$

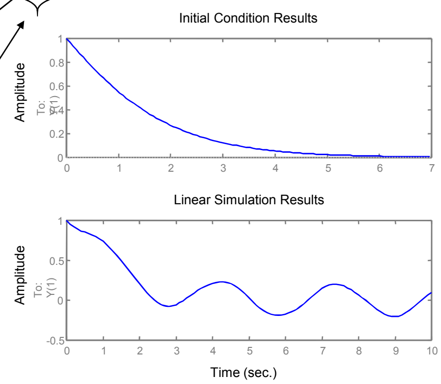
# Matlab Tools for Linear Systems

$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

```
A = [-1 1; 0 -1];  B = [0; 1];
C = [1 0]; D = [0];
x0 = [1; 0.5];

sys = ss(A,B,C,D);
initial(sys, x0);
impulse(sys);

t = 0:0.1:10;
u = 0.2*sin(5*t) + cos(2*t);
lsim(sys, u, t, x0);
```



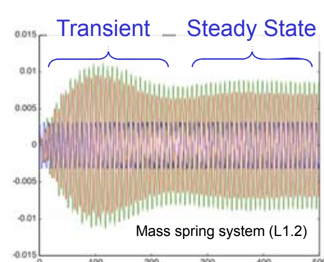

- Other MATLAB commands
  - gensig, square, sawtooth – produce signals of diff. types
  - step, impulse, initial, lsim – time domain analysis
  - bode, freqresp, evalfr – frequency domain analysis

ltiview – linear time invariant system plots



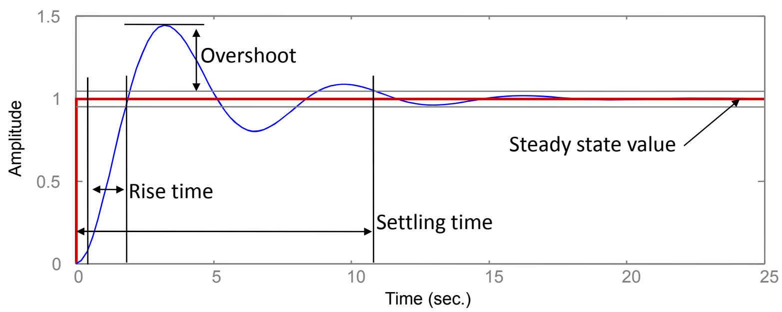
# Input/Output Performance

- Return to system with inputs
  - How does system respond to changes in input values?
- Transient response:
  - What happens right after a new input is applied
- Steady state response:
  - What happens a long time after the input is applied
- Stability vs input/output performance
  - Systems that are close to instability typically exhibit poor input/output performance
  - Nearly unstable systems (slow convergence) often exhibit “ringing” (highly oscillatory response to [non-periodic] inputs)



# Step Response

- Output characteristics in response to a "step" input
  - Rise time: time required to move from 5% to 95% of final value
  - Overshoot: ratio between amplitude of first peak and steady state value
  - Settling time: time required to remain w/in *p*% (usually 2%) of final value
  - Steady state value: final value at t = ∞

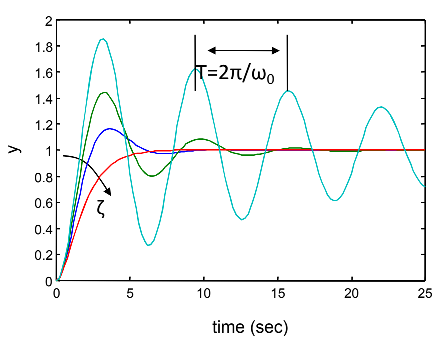





# Second Order Systems

- If you understand response of first and second order systems, you understand the response for any order (eig(A) are either real or complex)
  - Exception is non-diagonalizable A (non-trivial Jordan form)

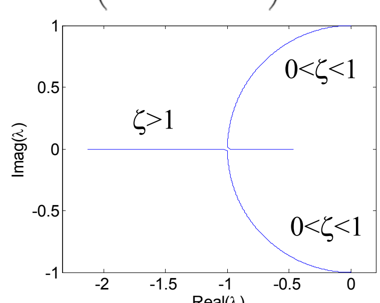
$$\ddot{q}+2\zeta\omega_0\dot{q}+\omega_0^2 q = u \quad \leftrightarrow \quad \frac{d}{dt}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\omega_0^2 & -2\zeta\omega_0\end{bmatrix}x+\begin{bmatrix}0\\1\end{bmatrix}u$$




- Analytical formulas exist for overshoot, rise time, settling time, etc

For $\zeta < 1$, eigenvalues at

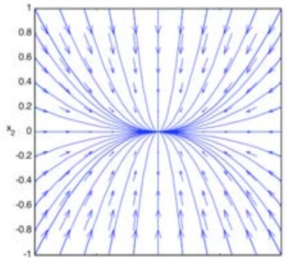
$$\left(-\zeta \pm j\sqrt{1-\zeta^2}\right)\omega_0$$



# Eigenstructure of Linear Systems

Real e-values

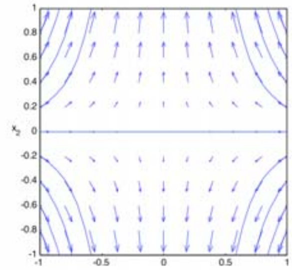
$Re(\lambda_i) < 0$



Real e-values

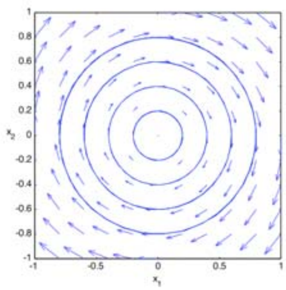
$Re(\lambda_i) < 0$

$Re(\lambda_i) > 0$



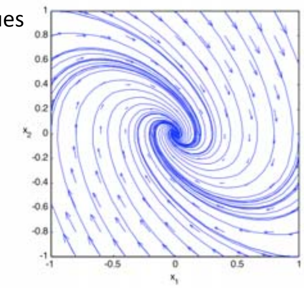
Complex e-values

$Re(\lambda_i) = 0$



Complex e-values

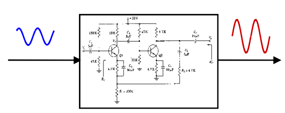
$Re(\lambda_i) < 0$



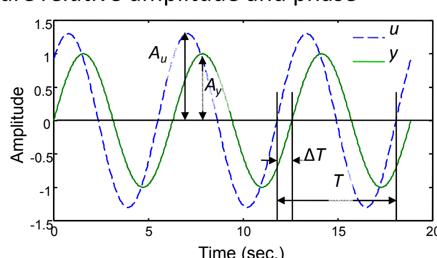


# Frequency Response

- Measure the *steady state* response of the system to sinusoidal input
  - Example: audio amplifier – would like consistent ("flat") amplification between 20 Hz & 20,000 Hz
  - Individual sinusoids are good *test signals* for measuring performance in many systems (e.g., seasonal cycles in temperature)



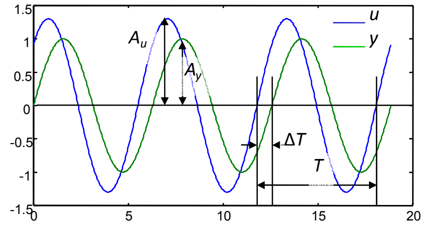
- Approach: plot input and output, measure *relative* amplitude and phase
  - Use MATLAB or SIMULINK to generate response of system to sinusoidal output
  - Gain = $A_y/A_u$
  - Phase = $2\pi \cdot \Delta T/T$




- May not work for *nonlinear* systems
  - System nonlinearities can cause *harmonics* to appear in the output
  - Amplitude and phase may not be well-defined
  - For *linear* systems, frequency response is always well defined

## Computing Frequency Responses

- Technique #1: plot input and output, measure relative amplitude and phase
  - Use MATLAB or SIMULINK to generate response of system to sinusoidal output
  - Gain = $A_y/A_u$
  - Phase = $2\pi \cdot \Delta T/T$
  - For *linear* system, gain and phase don't depend on the input amplitude



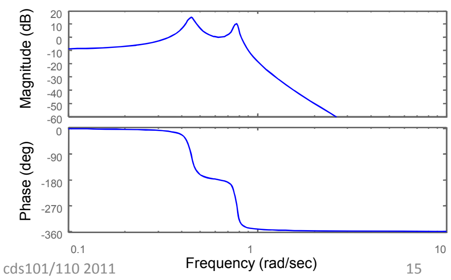

- Technique #2 (linear systems): use MATLAB bode command

bode(ss(A,B,C,D))

  - Assumes linear dynamics in state space form:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

  - Gain plotted on log-log scale
    - dB = 20 $\log_{10}$ (gain)
  - Phase plotted on linear-log scale

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## Calculating Frequency Response from convolution equation… (more later)

- Convolution equation describes response to any input; use this to look at response to sinusoidal input: $u(t) = A\sin(\omega t) = \frac{A}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$

$u$(t)

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B e^{i\omega\tau} d\tau$$
$$= e^{At}x(0) + e^{At}(i\omega I - A)^{-1} e^{(i\omega I - A)\tau}\Big|_{\tau=0}^{t} B$$
$$= e^{At}x(0) + e^{At}(i\omega I - A)^{-1}\left(e^{(i\omega I - A)t} - I\right)B$$
$$= e^{At}\left(x(0) - (i\omega I - A)^{-1}B\right) + (i\omega I - A)^{-1}Be^{i\omega t}$$

Transient (decays if stable) — Ratio of response/input

$$y(t) = Cx(t) + Du(t)$$
$$= Ce^{At}\left(x(0) - (i\omega I - A)^{-1}B\right) + \left(C(i\omega I - A)^{-1}B + D\right)e^{i\omega t}$$

"Frequency response"

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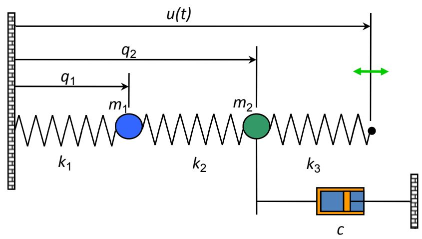
## Spring Mass System



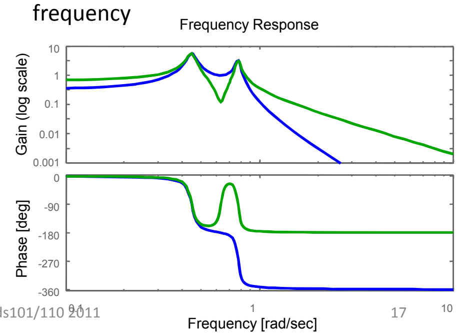

$$\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m} & \frac{k_2}{m} & 0 & 0 \\ \frac{k_2}{m} & -\frac{k_2+k_3}{m} & 0 & -\frac{c}{m} \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

With $k_1 = k_2 = 1,\ m = 1,\ c = 0$

$$v_{1,2} = \begin{bmatrix} 1 \\ 1 \\ \pm 1i \\ \pm 1i \end{bmatrix} \qquad v_{3,4} = \begin{bmatrix} 1 \\ -1 \\ \pm\sqrt{2}i \\ \mp\sqrt{2}i \end{bmatrix}$$

**Eigenvalues of A:**

- For zero damping, $\pm j\omega_1$ and $\pm j\omega_2$
- $\omega_1$ and $\omega_2$ correspond to the two peaks in the frequency response
- The eigenvectors for these eigenvalues give the *mode shape*:
  - In-phase motion for the lower frequency
  - Out-of phase motion for the higher frequency






## Frequency Response of the Climate

- Sinusoidal variation of radiative forcing, evaluate frequency response
- One- or two-reservoir energy balance models do a poor job, e.g.

$$C\frac{\mathrm{d}T}{\mathrm{d}t} = F - \lambda T \quad \Rightarrow \quad H_1(s) = \frac{1}{Cs + \lambda}$$

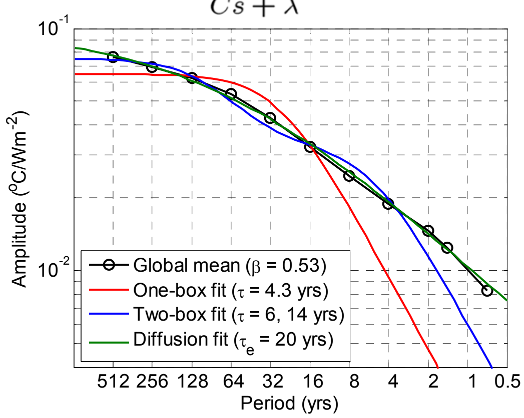
- Global mean temperature response consistent with diffusion into semi-infinite medium:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial z^2}$$

$$\kappa \left.\frac{\partial T}{\partial z}\right|_{z=0} = F - \lambda T(t, 0)$$

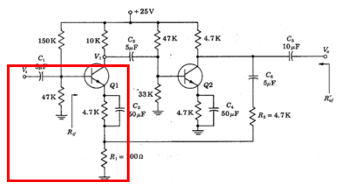
$$\Downarrow$$

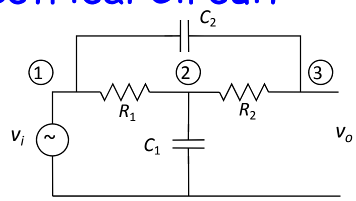
$$H_D(s) = \frac{1}{\lambda + \kappa (s/\alpha)^{1/2}}$$




MacMynowski, Shin, Caldeira, *Geoph. Res. Lett*, 2011.
http://www.youtube.com/watch?v=PJv4zL1HR70

# Example: Electrical Circuit






"Bridged Tee Circuit"

- Derivation based on Kirchoff's laws for electrical circuits (Ph 2)
  - Sum of currents at nodes = 0:

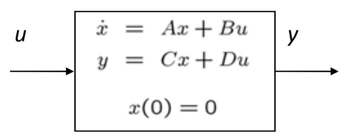
$$C_1 \frac{dv_2}{dt} = \frac{v_1 - v_2}{R_1} - \frac{v_2 - v_3}{R_2} \qquad C_2 \frac{d(v_3 - v_1)}{dt} = -\frac{v_3 - v_2}{R_2}$$

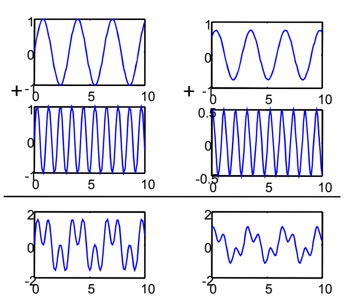
  - Rewrite in terms of new states: $v_{c1}=v_2$, $v_{c2}=v_3 - v_1$

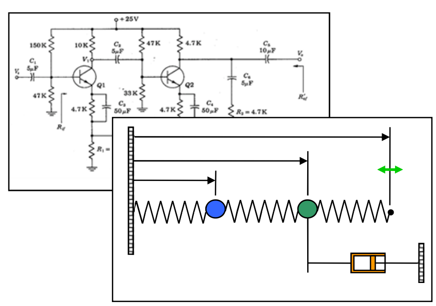
$$\frac{d}{dt}\begin{bmatrix} v_{c1} \\ v_{c2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) & -\frac{1}{C_1 R_2} \\ -\frac{1}{C_2 R_2} & -\frac{1}{C_2 R_2} \end{bmatrix} \begin{bmatrix} v_{c1} \\ v_{c2} \end{bmatrix} + \begin{bmatrix} \frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) \\ V_{c2} \end{bmatrix} v_i \qquad v_o = \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} v_{c1} \\ v_{c2} \end{bmatrix} + v_i$$



# Summary: Linear Systems





$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

- Properties of linear systems
  - Linearity with respect to initial condition and inputs
  - Stability characterized by eigenvalues
  - Many applications and tools available
  - Provide local description for nonlinear systems