Classical Control
Design Guidelines & Tools (L10.1)

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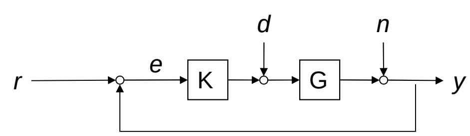
- Summarize frequency-domain control design guidelines and approach

Transfer Functions



$$\frac{e}{r} = \frac{1}{1+GK} = S$$ Tracking error (Sensitivity)

$$\frac{y}{r} = \frac{GK}{1+GK} = T$$ Tracking response (Complementary Sensitivity)

$$\frac{u}{r} = \frac{K}{1+GK} = KS$$ Actuator response

$$\frac{e}{d} = \frac{G}{1+GK} = GS$$ Disturbance response

$$\frac{(y/d)_{CL}}{(y/d)_{OL}} = S$$ Improvement in disturbance rejection

Design Guidelines

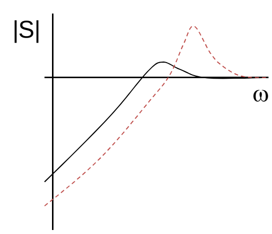
1. Design the loop transfer function L=GK, not the closed-loop
   a) L large: good performance
   b) L small: good robustness

2. Steady-state error based on L(0)
   a) Zero steady-state error to step requires integrator in L(s)
   b) Zero steady-state error to ramp requires two integrators
   c) Note that more than two integrators is harder
      - Conditionally stable (finite negative gain margin)

3. Cross-over with slope of -1
   = 0 then there is no crossover
   = -1 then phase is -90°
   = -2 then phase is -180° ⇒ Zero phase margin
   a) Behaviour near crossover is what influences stability
   b) Usual problem is *losing* phase (higher order dynamics, filters, time delays,…)

Guidelines, cont'd

4. Phase margin required for both robustness and performance
   a) Typically want 30° < PM < 60°
   (30 is typically absolute minimum,
   generally no advantage to more than 60)
   b) Can trade bandwidth for phase margin
      - Higher bandwidth → faster response
      - Lower phase margin → worse overshoot



Guidelines, cont'd

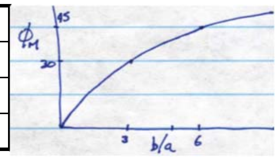
5. Main design tools:
   a) Lead: $K(s) = \frac{s+a}{s+b}, \qquad a < b$

   Adds phase, maximum phase ϕ_m added at

   $\omega = \sqrt{ab} \qquad \phi_m = 90° - 2\tan^{-1}\sqrt{a/b}$

   E.g.

ϕ_m	b/a
30°	~3
45°	~6
60°	~14



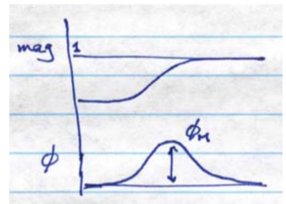
   or $K(s) = \frac{\alpha T s + 1}{Ts + 1}$

   a) Lag:

   $K(s) = \frac{s+a}{s+b}, \qquad b < a < \omega_c$

   a/b = increase in error constant

   Use for steady-state performance



c) PID:

$$K(s) = k_i/s + k_p + k_d s = \frac{s^2 k_d + s k_p + k_i}{s}$$

- May not need all three terms
- For second order system $G(s)=(s^2 + 2\zeta\omega s + \omega^2)^{-1}$ need derivative term to ensure slope is -1 if crossover is above ω
- For simple system G(s) = 1, then integral gain may suffice

Guidelines, cont'd

6. Bandwidth constrained by RHP zero:

$$G(s) = -\frac{s-a}{(s+b)^2} = \left[\frac{s+a}{(s+b)^2}\right] \cdot \left[-\frac{s-a}{s+a}\right] = G_{mp} \cdot G_{ap}$$

Minimum-phase, same magnitude as G

"all-pass": unity magnitude, pure phase lag. E.g. at s=ja/2, $\Delta\phi$ = 30° + 30° = 60°

Also note that S(a) = 1

- Maximum bandwidth for non-minimum phase systems
- Minimum bandwidth requirement for unstable systems

7. Time delay $= e^{-s\tau} \simeq -\frac{s-a}{s+a}, \qquad a = 2/\tau$

- Phase lag = 360° at f = $1/\tau$ (one full cycle)
  = 36° at f = 1/10 of $1/\tau$
- 1st order Padé error only ~1° at f=$0.1/\tau$ (or 4° at $\omega=1/\tau$)

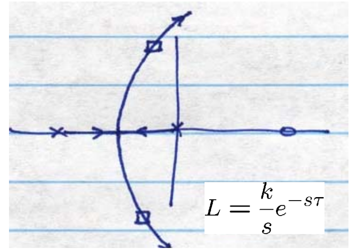
Guidelines, cont'd

8. Closed-loop performance typically looks close to a 2nd order system
   a) There is typically a dominant pair of poles that limit ability to increase loop gain any further
   - These are typically associated with phase margin at crossover:

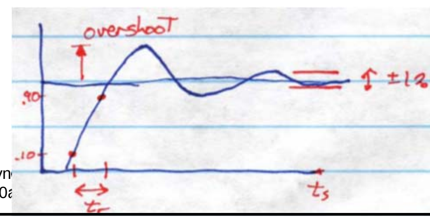
$$S(s) \simeq \frac{n(s)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

   - where ω_n ~ loop crossover frequency
     $\zeta \sim \phi_m/100$ (phase margin in degrees)

   b) Step response can be approximated (but use Matlab for accuracy)
      $t_r \sim 2.2\tau$ (where $\tau = 1/\omega_c$)
      $t_s \sim 4\tau$ if critically damped
      $t_s \sim 4/(\zeta\omega_n)$
      Overshoot $P \sim 1+e^{-\pi/\tan\theta}$, $\theta = \cos^{-1}\zeta$
      (e.g. PM=30°, P=1.37)



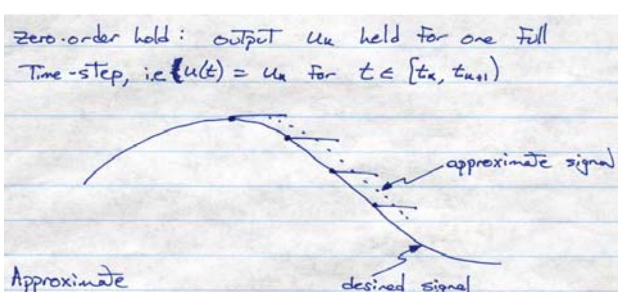
k = 0.5	k = 1	k = 1.5
ϕ_m= 62°	= 37°	= 16°
ζ = 0.75	= 0.35	= 0.14
ω_c = 0.5	= 1	= 1.5
ω_n = 1	= 1.4	= 1.7



Guidelines…

9. Digital implementation:
   a) Add time delay to plant model
   b) Design in continuous-time
   c) Convert to discrete-time to implement
   d) Approximate ZOH by delay of T/2,
      Include compute delay
      Include anti-aliasing and reconstruction filters

OK if $f_s > 10f_c$ where f_s is the sample rate and f_c is loop cross-over frequency



Converting to discrete-time: use "c2d" with "tustin" option

Cont'd…

10. Actuator saturation: $\frac{u}{r} = \frac{K}{1+GK}$
    a) If |GK| small, should keep |K| small also
    b) If saturating within control bandwidth, then $u=u_{sat}$ probably best anyway
    This effectively reduces the gain! Be careful with conditionally stable control…
    c) Be careful about integrator windup

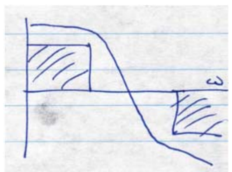
11. MIMO: via sequential loop closure only
    a) Associate actuators with sensors
    b) E.g. aircraft flight control: $\begin{bmatrix} \text{climb rate} \\ \text{velocity} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} \text{elevator} \\ \text{thrust} \end{bmatrix}$

    Close one loop
    (e.g. for system from thrust to velocity)
    and design second loop (e.g. elevator to climb rate)
    with first loop closed (Works fine if $G_{21}=0$)

12. Limits on performance, e.g. S + T = 1, $\int_0^\infty \log|S(j\omega)|d\omega = \pi \sum \mathrm{Re}(p_k) \geq 0$

Design Approach

1. Convert performance specifications into specifications on L(s) $\left|\frac{e}{r}\right| < \frac{1}{a} \Rightarrow |L| > a$



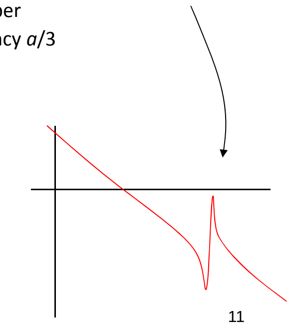
2. Add integrator(s) if needed for steady-state performance

3. Estimate crossover frequency
    a) Sufficient to meet specification (e.g. if need 10% error at freq *x*, crossover will be ~10*x*)
    b) Specification is often "as good as possible":
        i. If bandwidth limited by phase lag (frequent), construct phase budget:
            – crossover slope of -1 gives 90° phase
            – Lead of 30° is easy to add
            – For 60° phase margin, can tolerate 60° phase lag due to time delay, non-minimum phase part of plant, actuator lag (minimum phase, but inverting will typically lead to saturation)
        ii. May be limited by actuator saturation, model uncertainty, or sensor noise…

Design Approach, cont'd

4. Crossover at slope of -1
    a) Choose appropriate number and frequency of poles and zeros (invert minimum-phase part of plant)… same as PID design knobs

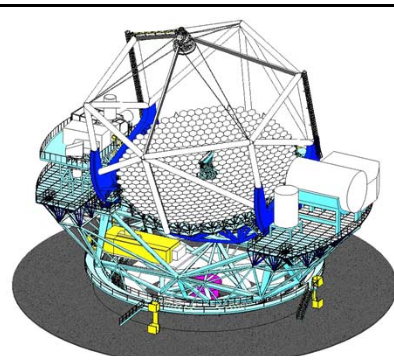
5. Clean up
    a) Add lead to fix phase at cross-over and obtain desired phase margin
    b) Add lag at low frequencies if need to boost low frequency gain
    c) Add notch filters to deal with problem high-frequency lightly-damped modes and recover gain margin
    d) Add roll-off to ensure compensator is proper or strictly proper
        - Single pole at frequency *a* gives 18° phase lag at frequency *a*/3
    e) Adjust lead (and maybe bandwidth) to compensate for phase lag added in steps (b)-(d)

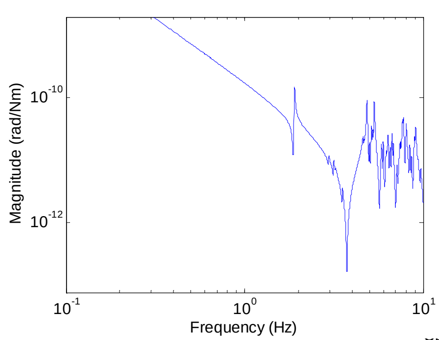


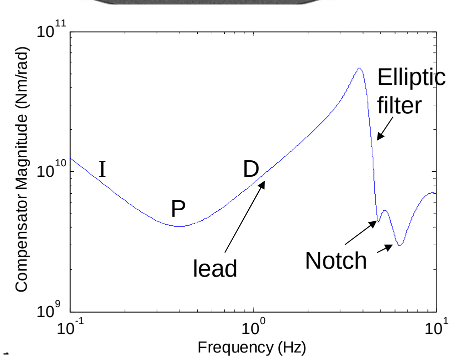
6. Plot "gang of four" frequency and step response
7. Simulate and iterate
8. Test and iterate

Example

- Thirty meter telescope pointing control
    - Input is drive torque about elevation axi
    - Output is (collocated) rotation from encoder output
    - Need notch filters to compensate for resonant peaks



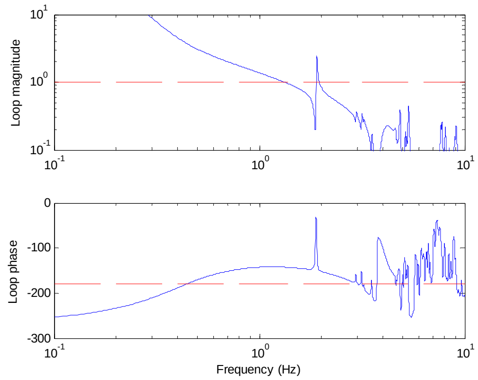


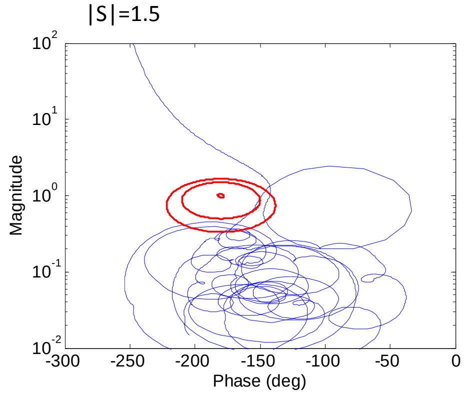


Loop Transfer Function

- Loop crossover frequency = 1.3 Hz
- Bandwidth: -3dB sensitivity = 0.63 Hz
- 37° phase margin, 6 dB gain margin

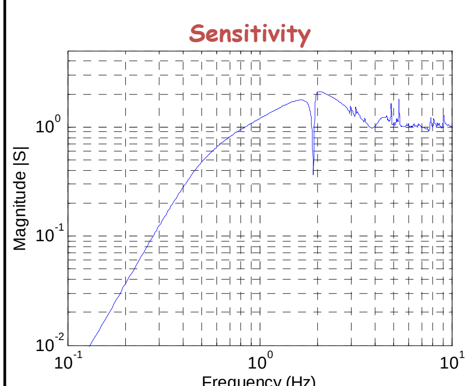
- Nichols chart: Same as Nyquist, but log-magnitude and phase instead of real/imag
- Red lines plotted at |S|=2 and |S|=1.5

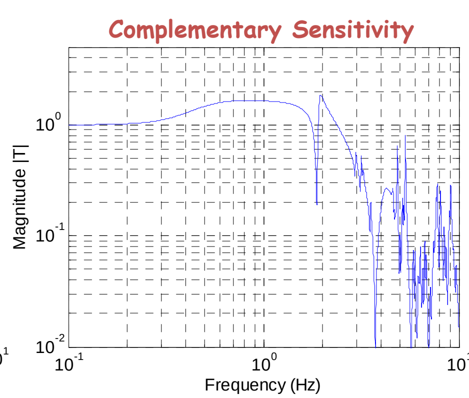




Performance

- Good disturbance rejection up to ~0.5 Hz
- Good tracking performance up to ~0.3 Hz





Summary

- Loop shaping (frequency-domain) design provides intuition about what can be achieved and how to achieve it
  - Convert performance specifications into specification on L(s)
  - Figure out how to stabilize unstable poles (e.g. with Nyquist plot)
  - Figure out required bandwidth
  - Cross-over with slope of –1
  - Add phase if needed
  - Plot gang of 4
  - If uncertain about stability, plot Nyquist.