CDS 101: Lecture 4-1
Reachability and State Space Feedback

Douglas G. MacMynowski
18 October 2010

Goals:

Define reachability of a control system

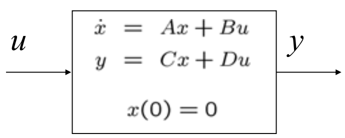
Give tests for reachability of linear systems and apply to examples

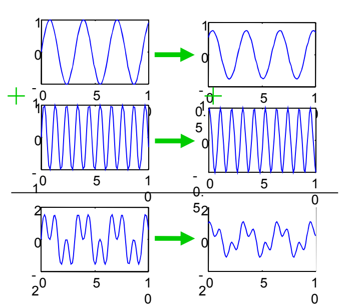
Describe the design of state feedback controllers for linear systems

Reading:

Åström and Murray, *Feedback Systems*, Ch 6

Review from last week

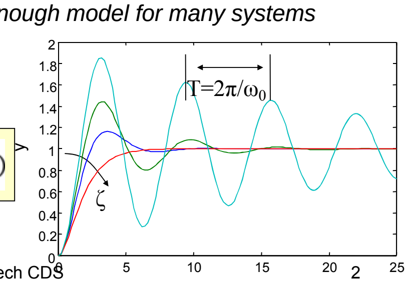




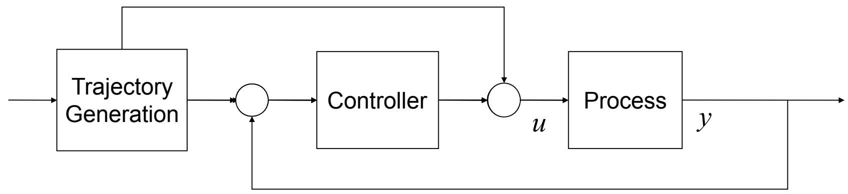
$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau + Du(t)$$

Properties of Linear Systems:

- Linearity with respect to inputs and initial conditions
- Stability characterized by eigenvalues
- Response described by convolution integral
- Many applications and tools
- Characterizes a nonlinear system around an equilibrium point
  - *No system is linear for all input amplitudes, but linearization is a good enough model for many systems*



Control Overview



Design controller so that

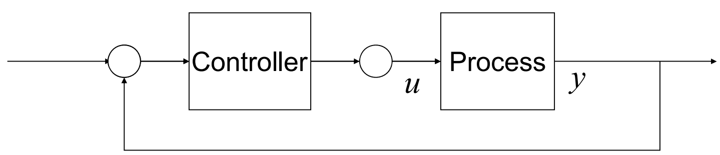
i) System is stable

ii) Performance:

- Keep close to equilibrium despite disturbances (disturbance rejection)
- Move system to desired state (track reference)

iii) Robustness to modeling errors

First Questions…



1) Can the input *u* affect the dynamics?

e.g.
$$\begin{aligned}\dot{x}_1 &= x_1 + u \\ \dot{x}_2 &= x_2\end{aligned}$$
⇒ Can't change x_2

Equivalent to asking whether there is a *u* that allows us to *reach* any point in the state-space

⇒ Reachability (today), depends on A, B

⇒ Related to the design of state feedback

2) Does the measurement *y* contain enough information about the system?

e.g.
$$\begin{aligned}\dot{x}_1 &= x_1 \qquad y = x_1 \\ \dot{x}_2 &= x_2\end{aligned}$$
⇒ Can't measure x_2

⇒ Observability (next week), depends on A, C

⇒ Related to the design of observers to estimate state from measurement

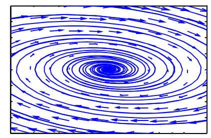
Control Design Concepts

System description: single input, single output system (MIMO also OK)

$$\begin{aligned} \dot{x} &= f(x,u) \quad & x \in \mathbb{R}^n,\ x(0) \text{ given} \\ y &= h(x,u) \quad & u \in \mathbb{R},\ y \in \mathbb{R} \end{aligned}$$

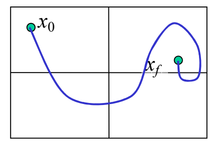
Stability: stabilize the system around an equilibrium point

- Given equilibrium point $x_e \in \mathbb{R}^n$, find control "law" $u = \alpha(x)$ such that $\lim_{t\to\infty} x(t) = x_e \, \forall x(0) \in \mathbb{R}^n$



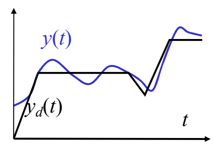
Reachability: steer the system between two points

- Given $x_0, x_f \in \mathbb{R}^n$ find an input $u(t)$ such that $\dot{x} = f(x, u(t))$ takes $x(t_0) = x_0 \rightarrow x(T) = x_f$



Tracking: track a given output trajectory

- Given $y_d(t)$, find $u = \alpha(x,t)$ such that $\lim_{t\to\infty} (y(t) - y_d(t)) = 0,\ \forall x(0) \in \mathbb{R}^n$



Reachability of Input/Output Systems

$$\begin{aligned} \dot{x} &= f(x,u) \quad & x \in \mathbb{R}^n,\ x(0) \text{ given} \\ y &= h(x,u) \quad & u \in \mathbb{R},\ y \in \mathbb{R} \end{aligned}$$

Defn An input/output system is *reachable* if for any $x_0, x_f \in \mathbb{R}^n$ and any time $T > 0$ there exists an input $u_{[0,T]} \in \mathbb{R}^n$ such that the solution of the dynamics starting from $x(0)=x_0$ and applying input $u(t)$ gives $x(T)=x_f$.

Remarks

- In the definition, x_0 and x_f do not have to be equilibrium points $\Rightarrow$ we don't necessarily stay at x_f after time T.
- Reachability is defined in terms of states $\Rightarrow$ doesn't depend on output
- For *linear systems,* can characterize reachability by looking at the general solution:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \qquad x(T) = e^{AT}x_0 + \int_{\tau=0}^{T} e^{A(T-\tau)}Bu(\tau)d\tau$$

Tests for Reachability

$$\begin{aligned}\dot{x} &= Ax + Bu \quad x \in \mathbb{R}^n,\ x(0) \text{ given}\\ y &= Cx + Du \quad u \in \mathbb{R},\ y \in \mathbb{R}\end{aligned} \qquad x(T) = e^{AT}x_0 + \int_{\tau=0}^{T} e^{A(T-\tau)}Bu(\tau)\mathrm{d}\tau$$

Thm A linear system is reachable if and only if the $n \times n$ *reachability matrix*

$$\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

is full rank.

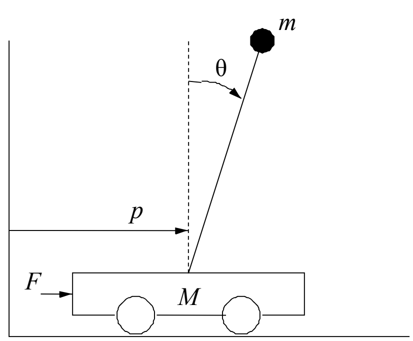
Remarks

- *Very* simple test to apply. In MATLAB, use ctrb(A,B) and check rank w/ det()
- If this test is satisfied, we say “the pair (A,B) is reachable”
- Some insight into the proof can be seen by expanding the matrix exponential

$$\begin{aligned} e^{A(T-\tau)}B &= \left(I + A(T-\tau) + \frac{1}{2}A^2(T-\tau)^2 + \cdots + \frac{1}{(n-1)!}A^{n-1}(T-\tau)^{n-1} + \cdots\right)B \\ &= B + AB(T-\tau) + \frac{1}{2}A^2B(T-\tau)^2 + \cdots + \frac{1}{(n-1)!}A^{n-1}B(T-\tau)^{n-1} + \cdots \end{aligned}$$

- Test does not give a measure of how much control effort is required
- Other tests for reachability also exist

Example #1: Linearized pendulum on a cart



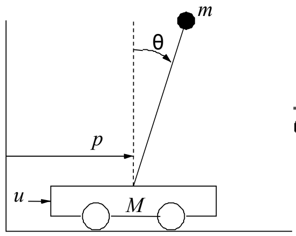
Question: can we locally control the position of the cart by proper choice of input?

Approach: look at the linearization around the upright position (good approximation to the full dynamics if θ remains small)

$$\frac{d}{dt}\begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2l^2g}{M_tJ_t - m^2l^2} & \frac{-cJ_t}{M_tJ_t - m^2l^2} & \frac{-\gamma J_t lm}{M_tJ_t - m^2l^2} \\ 0 & \frac{M_tmgl}{M_tJ_t - m^2l^2} & \frac{-clm}{M_tJ_t - m^2l^2} & \frac{-\gamma M_t}{M_tJ_t - m^2l^2} \end{bmatrix} \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{J_t}{M_tJ_t - m^2l^2} \\ \frac{lm}{M_tJ_t - m^2l^2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x,$$

Example #1, con't: Linearized pendulum on a cart



$$\frac{d}{dt}\begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2 l^2 g}{\mu} & \cancelto{0}{\frac{-cJ_t}{\mu}} & \cancelto{0}{\frac{-\gamma J_t l m}{\mu}} \\ 0 & \frac{M_t m g l}{\mu} & \cancelto{0}{\frac{-c l m}{\mu}} & \cancelto{0}{\frac{-\gamma M_t}{\mu}} \end{bmatrix} \begin{bmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{J_t}{\mu} \\ \frac{lm}{\mu} \end{bmatrix} u$$

$$\mu = M_t J_t - m^2 l^2$$

- Simplify by setting c, $\gamma = 0$

Reachability matrix

$$W_r = \left[\begin{array}{c|c|c|c} 0 & \frac{J_t}{\mu} & 0 & \frac{g l^3 m^3}{\mu^2} \\ 0 & \frac{lm}{\mu} & 0 & \frac{g l^2 m^2 (m+M)}{\mu^2} \\ \frac{J_t}{\mu} & 0 & \frac{g l^3 m^3}{\mu^2} & 0 \\ \frac{lm}{\mu} & 0 & \frac{g l^2 m^2 (m+M)}{\mu^2} & 0 \\ B & AB & A^2B & A^3B \end{array}\right].$$

- Full rank as long as constants are such that columns 1 and 3 are not multiples of each other
- Reachable as long as

$$\det(W_r) = \frac{g^2 \ell^4 m^4}{\mu^4} \neq 0$$

- Can "steer" linearization between points by proper choice of input

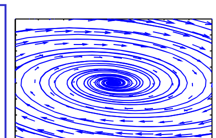
Control Design Concepts

System description: single input, single output system (MIMO also OK)

$$\begin{aligned} \dot{x} &= f(x,u) \qquad x \in \mathbb{R}^n,\ x(0) \text{ given} \\ y &= h(x,u) \qquad u \in \mathbb{R},\ y \in \mathbb{R} \end{aligned}$$

Stability: stabilize the system around an equilibrium point

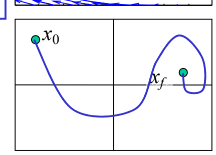
- **Given equilibrium point $x_e \in \mathbb{R}^n$, find control "law" u=α(x) such that** $\lim_{t\to\infty} x(t) = x_e,\ \forall x(0) \in \mathbb{R}^n$



✓ **Reachability: steer the system between two points**

- **Given $x_0, x_f \in \mathbb{R}^n$, find an input u(t) such that**

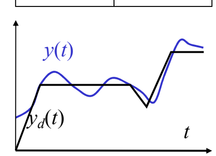
$$\dot{x} = f(x, u(t)) \text{ takes } x(t_0) = x_0 \rightarrow x(T) = x_f$$



Tracking: track a given output trajectory

- **Given $y_d(t)$, find u=α(x,t) such that**

$$\lim_{t\to\infty} (y(t) - y_d(t)) = 0,\ \forall x(0) \in \mathbb{R}^n$$



State space controller design for linear systems

$$\dot{x} = Ax + Bu \quad x \in \mathbb{R}^n,\ x(0) \text{ given}$$
$$y = Cx + Du \quad u \in \mathbb{R},\ y \in \mathbb{R}$$

$$x(T) = e^{AT}x_0 + \int_{\tau=0}^{T} e^{A(T-\tau)}Bu(\tau)d\tau$$

Goal: find a linear control law $u = -Kx$ such that the closed loop system

$$\dot{x} = Ax - BKx = (A - BK)x$$

is stable at $x_e = 0$.

Remarks

- Stability based on eigenvalues ⇒ use K to make eigenvalues of $(A - BK)$ stable
- Can also link eigenvalues to *performance* (e.g., initial condition response)
- Question: when can we place the eigenvalues any place that we want?

Theorem The eigenvalues of $(A - BK)$ can be set to arbitrary values if and only if the pair (A, B) is reachable.

MATLAB: K = place(A, B, eigs)

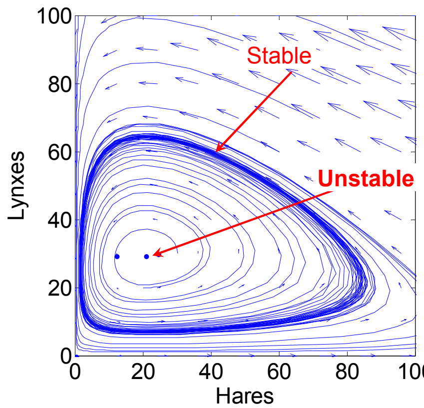
Example #2: Predator prey

Natural dynamics

$$\frac{dH}{dt} = rH\left(1 - \frac{H}{k_c}\right) - \frac{aHL}{c+H}, \qquad H \geq 0$$
$$\frac{dL}{dt} = b\frac{aHL}{c+H} - dL, \qquad L \geq 0$$

Controlled dynamics: modulate food supply

$$\frac{dH}{dt} = (r+u)H\left(1 - \frac{H}{k_c}\right) - \frac{aHL}{c+H}$$
$$\frac{dL}{dt} = b\frac{aHL}{c+H} - dL,$$



Q1: can we move from some initial population of lynxes and hares to a specified one in time T by modulation of the food supply?

Q2: can we *stabilize* the population around the desired equilibrium point

Approach: try to answer this question *locally*, around the natural equilibrium point

Example #2: Problem setup

Equilibrium point calculation

$$\frac{dH}{dt} = (r+u)H\left(1-\frac{H}{k_c}\right) - \frac{aHL}{c+H}$$
$$\frac{dL}{dt} = b\frac{aHL}{c+H} - dL,$$

- x_e = (20.5,29.5), u_e = 0

```
% Compute the equil point
% predprey.m contains dynamics
f = inline('predprey(0,x)');
xeq = fsolve(f, [20, 30]);

% Compute linearization
A = [...];
B = [H0*(1 - H0/K); 0];
p = [-1;-2];
K = place(A,B,p)
```

Linearization

- Compute linearization around equil. point, x_e:

$$A = \left.\frac{\partial f}{\partial x}\right|_{x_e,u_e} \qquad B = \left.\frac{\partial f}{\partial u}\right|_{x_e,u_e}$$

- Redefine local variables: $z = x - x_e$ $v = u - u_e$

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} r - \frac{2H_0 r}{k} - \frac{aL_0}{c+H_0} + \frac{aL_0H_0}{(c+H_0)^2} & -\frac{aH_0}{c+H_0} \\ baL_0\left(\frac{1}{c+H_0} - \frac{H_0}{(c+H_0)^2}\right) & ba\frac{H_0}{c+H_0} - d \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} H_0\left(1-\frac{H_0}{k}\right) \\ 0 \end{bmatrix} v$$

- Reachable? YES, if $ba \neq 0$ (check [B AB]) ⇒ can locally steer to any point

Example #2: Stabilization via eigenvalue assignment

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} r - \frac{2H_0 r}{k} - \frac{aL_0}{c+H_0} + \frac{aL_0H_0}{(c+H_0)^2} & -\frac{aH_0}{c+H_0} \\ baL_0\left(\frac{1}{c+H_0} - \frac{H_0}{(c+H_0)^2}\right) & ba\frac{H_0}{c+H_0} - d \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} H_0\left(1-\frac{H_0}{k}\right) \\ 0 \end{bmatrix} v$$

Control design:

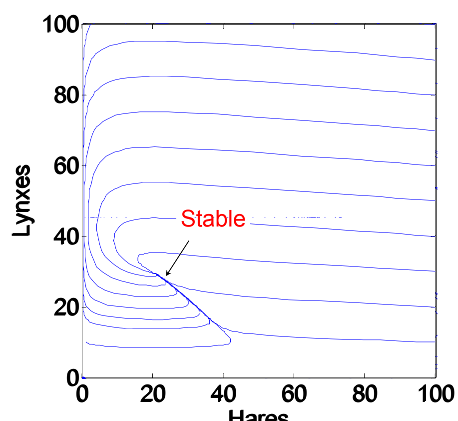
$$v = -Kz + k_r r$$
$$u = u_e - K(x - x_e) + k_r(r - y_e)$$

Place poles at stable values

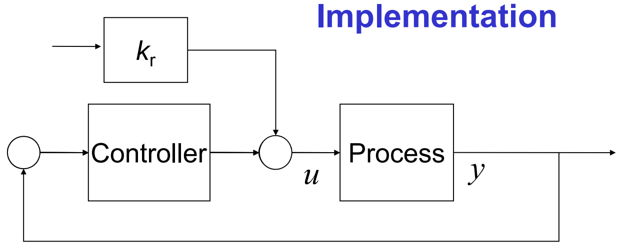
- Choose λ = -1, -2
- K = place(A, B, [-1; -2]);

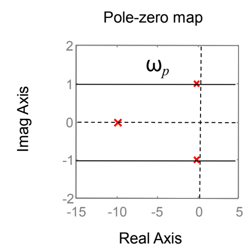
Modify NL dynamics to include control

$$\frac{dH}{dt} = (r+u)H\left(1-\frac{H}{k_c}\right) - \frac{aHL}{c+H}$$
$$\frac{dL}{dt} = b\frac{aHL}{c+H} - dL,$$



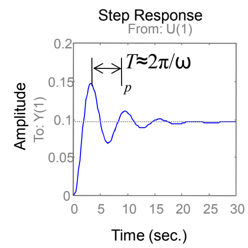
Implementation





Remarks:

1. In practice, don't always have access to full state
   → estimate state from measurement *y* (next week)
2. What to pick for eigenvalues?
   - For each eigenvalue $\lambda_i = \sigma_i + j\omega_i$, get contribution of the form $y_i(t) = e^{\sigma t}(a\sin(\omega t) + b\cos(\omega t))$
   - Faster response will require more control effort
   - Optimal control: LQR (in text, CDS 110b)



3. How to obtain desired tracking response so that $y_{ss} = r$ for some reference r?
   - Choose $u = -Kx + k_r r$
   - Steady state (if D=0): $y = Cx = C(A - BK)^{-1}Bk_r r \;\Rightarrow\; k_r = \text{inv}[C(A - BK)^{-1}B]$

Reachable Canonical Form

If the system is reachable, then there exists a transformation $z = Tx$ such that:

$$\dot{z} = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0\cdots & 0 & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & 1 & 0 \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u$$

(Check [B AB A^2B … A^{n-1}B]… is this system reachable?)

Characteristic polynomial is:

$$\lambda(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

Choose state feedback:

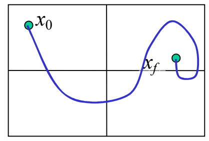
$$u = -Kz = -\begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} z$$

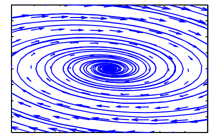
Then closed-loop system is:

$$\dot{z} = \begin{bmatrix} -a_1 - k_1 & -a_2 - k_2 & -a_3 - k_3 & \cdots & -a_n - k_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0\cdots & 0 & \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & 1 & 0 \end{bmatrix} z$$

Summary: Reachability and State Space Feedback

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$





$$\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

$$u = -Kx + k_r r$$

Key concepts

- Reachability: find u s.t. $x_0 \rightarrow x_f$
- Reachability rank test for linear systems
- State feedback to assign eigen-values