CDS 101/110a: Lecture 3.1
Linear Systems

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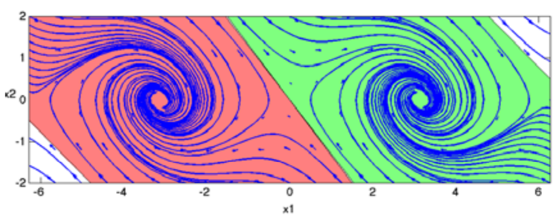
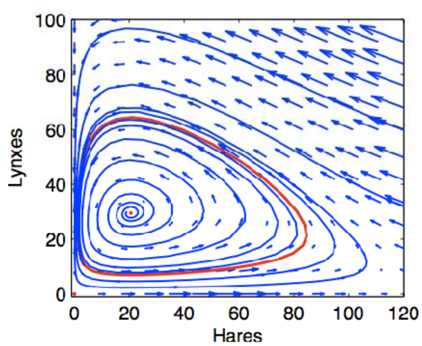
Goals:
- Describe linear system models: properties, examples, and tools
- Characterize stability and performance of linear systems in terms of eigenvalues
- Compute linearization of a nonlinear systems around an equilibrium point

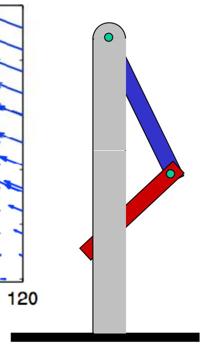
Reading:
- Åström and Murray, *Analysis and Design of Feedback Systems,* Ch 5

Review from Last Week



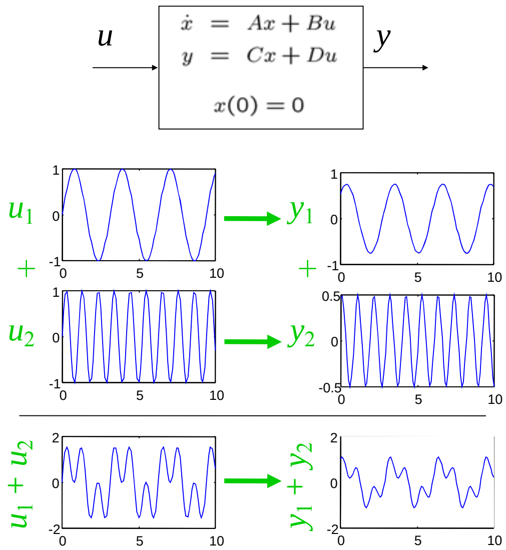
Key topics
- Stability of equilibrium points
- Eigenvalues determine stability for linear systems
- Local versus global behavior





- Note on eigenvalues: If eigenvectors are unique then:
  $$AV = V\Lambda$$
  $$\Rightarrow V^{-1}AV = \Lambda$$
- Choose: $x = V\eta$ then
  $$\dot{x} = Ax$$
  $$\Rightarrow \dot{\eta} = \Lambda\eta$$

Linear Systems



Input/output linearity at x(0) = 0

- Linear systems are linear in initial condition *and* input ⇒ need to use x(0) = 0 to add outputs together
- For different initial conditions, you need to be more careful

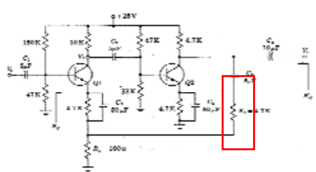
Linear system ⇒ step response and frequency response scale with input amplitude

- 2X input ⇒ 2X output
- Allows us to use *ratios* and *percentages* in step/freq response. These are *independent* of input amplitude
- Limitation: input saturation ⇒ only holds up to certain input amplitude

Why are Linear Systems Important?

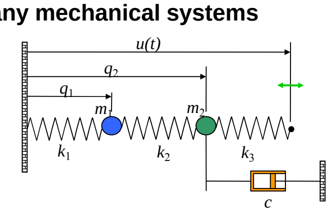
Many important *examples*

Electronic circuits



- Especially true after feedback
- Frequency response is key performance specification

Many mechanical systems



Quantum mechanics, Markov chains

Many important *tools*

Frequency response, step response, etc

- Traditional tools of control theory
- Developed in 1930's at Bell Labs

Classical control design toolbox — CDS 101/110a

- Nyquist plots, gain/phase margin
- Loop shaping

Optimal control and estimators — CDS 110b

- Linear quadratic regulators
- Kalman estimators

Robust control design — CDS 212

- H_∞ control design
- μ analysis for structured uncertainty

Solutions of Linear Systems: The Matrix Exponential

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned} \longrightarrow \quad y(t) = ???$$

Scalar linear system, with no input

$$\begin{aligned}\dot{x} &= ax \\ y &= cx\end{aligned} \quad x(0) = x_0 \longrightarrow x(t) = e^{at}x_0 \longrightarrow y(t) = ce^{at}x_0$$

Matrix version, with no input

$$\begin{aligned}\dot{x} &= Ax \\ y &= Cx\end{aligned} \quad x(0) = x_0 \longrightarrow x(t) = e^{At}x_0 \longrightarrow \boxed{y(t) = Ce^{At}x_0}$$

sys=ss(A,B,C,D);
initial(sys,x0);

Matrix exponential

- Analog to the scalar case; defined by series expansion:

$$e^M = I + M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \cdots$$

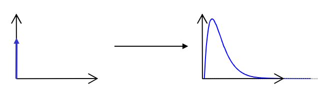
P = expm(M)

Linear Control Systems and Convolution

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned} \longrightarrow \quad y(t) = \underbrace{Ce^{At}x(0)}_{\text{homogeneous}} + ???$$

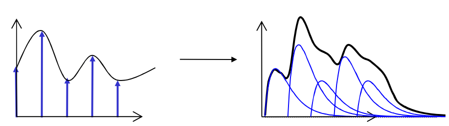
***Impulse response*, $h(t) = Ce^{At}B$**

- Response to input “impulse”
- Equivalent to “Green’s function”



Linearity $\Rightarrow$ compose response to arbitrary $u(t)$ using *convolution*

- Decompose input into “sum” of shifted impulse functions
- Compute impulse response for each
- “Sum” impulse response to find $y(t)$



Complete solution: use integral instead of “sum”

$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

- linear with respect to initial condition *and* input
- 2X input $\Rightarrow$ 2X output when $x(0) = 0$

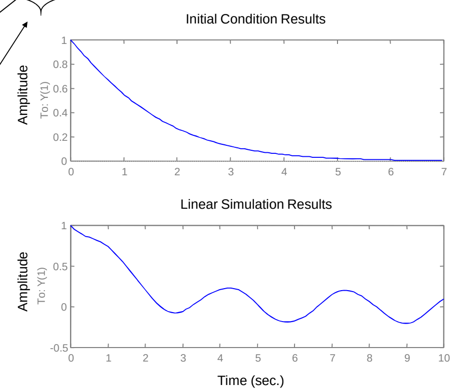
Matlab Tools for Linear Systems

$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

```
A = [-1 1; 0 -1];  B = [0; 1];
C = [1 0]; D = [0];
x0 = [1; 0.5];

sys = ss(A,B,C,D);
initial(sys, x0);
impulse(sys);

t = 0:0.1:10;
u = 0.2*sin(5*t) + cos(2*t);
lsim(sys, u, t, x0);
```



Other MATLAB commands

- gensig, square, sawtooth – produce signals of diff. types
- step, impulse, initial, lsim – time domain analysis
- bode, freqresp, evalfr – frequency domain analysis

ltiview – linear time invariant system plots

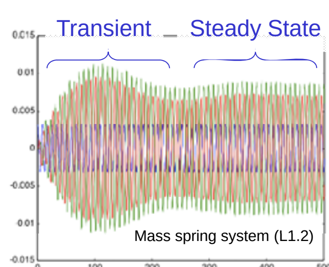
Input/Output Performance

Return to system with inputs

- How does system respond to changes in input values?

Transient response:

- What happens right after a new input is applied



Steady state response:

- What happens a long time after the input is applied

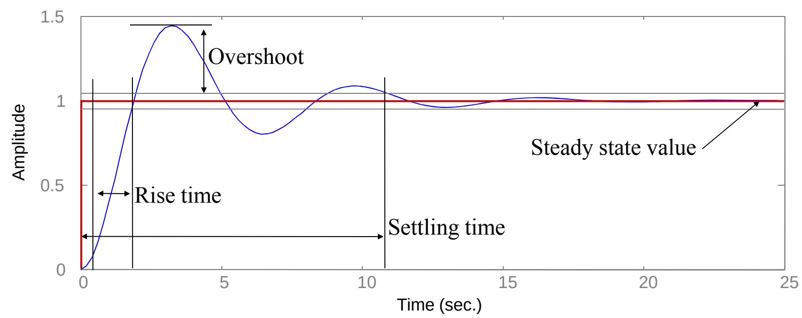
Stability vs input/output performance

- Systems that are close to instability typically exhibit poor input/output performance
- Nearly unstable systems (slow convergence) often exhibit "ringing" (highly oscillatory response to [non-periodic] inputs)

Step Response

Output characteristics in response to a "step" input

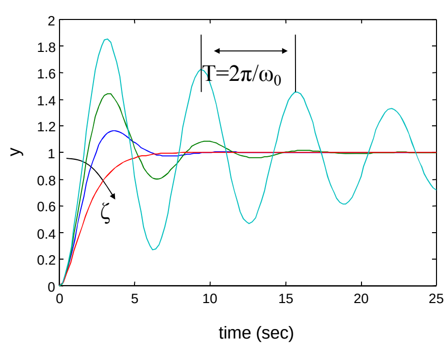
- Rise time: time required to move from 5% to 95% of final value
- Overshoot: ratio between amplitude of first peak and steady state value
- Settling time: time required to remain w/in *p*% (usually 2%) of final value
- Steady state value: final value at t = ∞



Second Order Systems

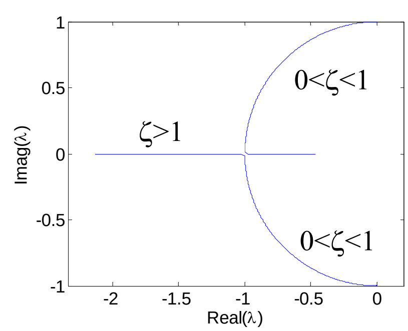
If you understand response of first and second order systems, you understand the response for any order (eig(A) are either real or complex)

$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = u \quad \leftrightarrow \quad \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$



For $\zeta < 1$, eigenvalues at

$$-\zeta \pm j\sqrt{1-\zeta^2}\,\omega_0$$

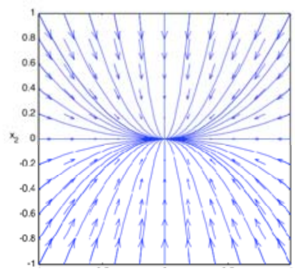


- Analytical formulas exist for overshoot, rise time, settling time, etc

Eigenstructure of Linear Systems

Real e-values

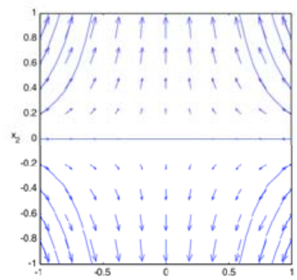
$\mathrm{Re}(\lambda_i) < 0$



Real e-values

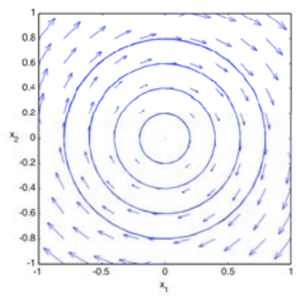
$\mathrm{Re}(\lambda_i) < 0$

$\mathrm{Re}(\lambda_i) > 0$



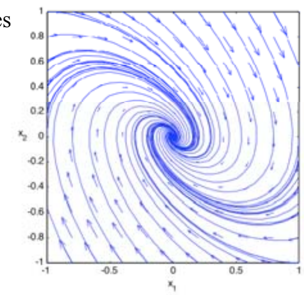
Complex e-values

$\mathrm{Re}(\lambda_i) = 0$



Complex e-values

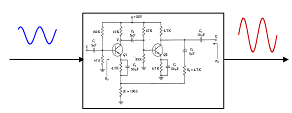
$\mathrm{Re}(\lambda_i) < 0$



Frequency Response

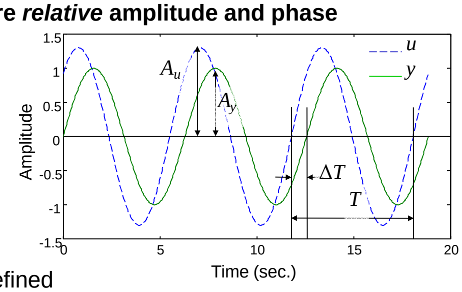
Measure the *steady state* response of the system to sinusoidal input

- Example: audio amplifier – would like consistent ("flat") amplification between 20 Hz & 20,000 Hz
- Individual sinusoids are good *test signals* for measuring performance in many systems (eg, seasonal cycles in temperature)



Approach: plot input and output, measure *relative* amplitude and phase

- Use MATLAB or SIMULINK to generate response of system to sinusoidal output
- Gain = A_y/A_u
- Phase = $2\pi \cdot \Delta T/T$



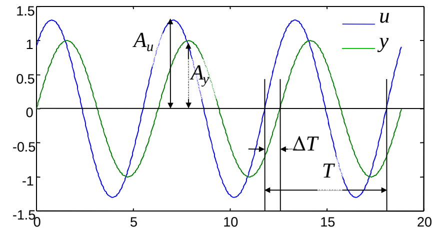
May not work for *nonlinear* systems

- System nonlinearities can cause *harmonics* to appear in the output
- Amplitude and phase may not be well-defined
- For *linear* systems, frequency response is always well defined

Computing Frequency Responses

Technique #1: plot input and output, measure relative amplitude and phase

- Use MATLAB or SIMULINK to generate response of system to sinusoidal output
- Gain = A_y/A_u
- Phase = $2\pi \cdot \Delta T/T$
- For *linear* system, gain and phase don't depend on the input amplitude



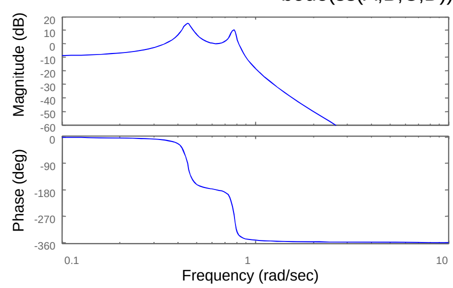
Technique #2 (linear systems): use MATLAB bode command

bode(ss(A,B,C,D))

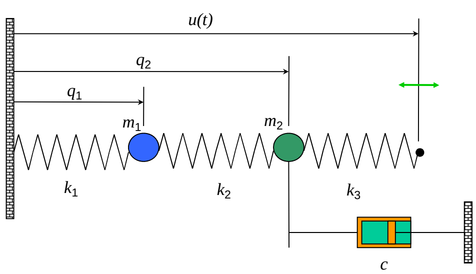
- Assumes linear dynamics in state space form:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

- Gain plotted on log-log scale
  - dB = 20 $\log_{10}$ (gain)
- Phase plotted on linear-log scale



Spring Mass System

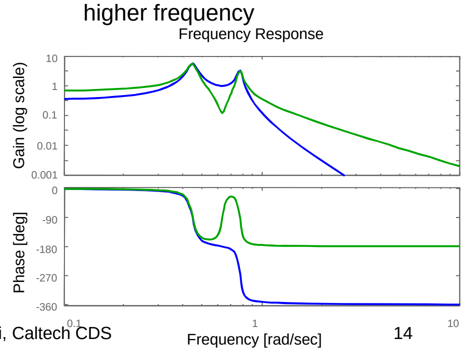


$$\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m} & \frac{k_2}{m} & 0 & 0 \\ \frac{k_2}{m} & -\frac{k_2+k_3}{m} & 0 & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

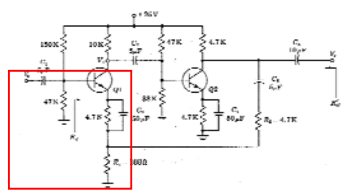
$$y = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

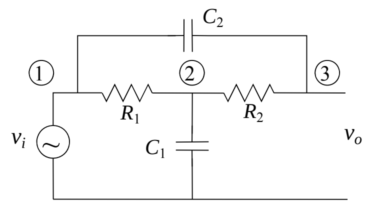
Eigenvalues of A:

- For zero damping, $\pm j\omega_1$ and $\pm j\omega_2$
- ω_1 and ω_2 correspond to the two peaks in the frequency response
- The eigenvectors for these eigenvalues give the *mode shape*:
  - In-phase motion for the lower frequency
  - Out-of phase motion for the higher frequency



Example: Electrical Circuit





"Bridged Tee Circuit"

Derivation based on Kirchoff's laws for electrical circuits (Ph 2)

- Sum of currents at nodes = 0:

$$C_1\frac{dv_2}{dt}=\frac{v_1-v_2}{R_1}-\frac{v_2-v_3}{R_2} \qquad C_2\frac{d(v_3-v_1)}{dt}=-\frac{v_3-v_2}{R_2}$$

- Rewrite in terms of new states: $v_{c1}=v_2$, $v_{c2}=v_3-v_1$

$$\frac{d}{dt}\begin{bmatrix}v_{c1}\\ v_{c2}\end{bmatrix}=\begin{bmatrix}-\frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) & -\frac{1}{C_1R_2}\\ -\frac{1}{C_2R_2} & -\frac{1}{C_2R_2}\end{bmatrix}\begin{bmatrix}v_{c1}\\ v_{c2}\end{bmatrix}+\begin{bmatrix}\frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\\ V_{c2}\end{bmatrix}v_i \qquad v_o=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}v_{c1}\\ v_{c2}\end{bmatrix}+v_i$$

Linearization Around an Equilibrium Point

$$\begin{aligned}\dot{x}&=f(x,u)\\ y&=h(x,u)\end{aligned} \Rightarrow \begin{aligned}\dot{z}&=Az+Bu\\ y&=Cz+Du\end{aligned}$$

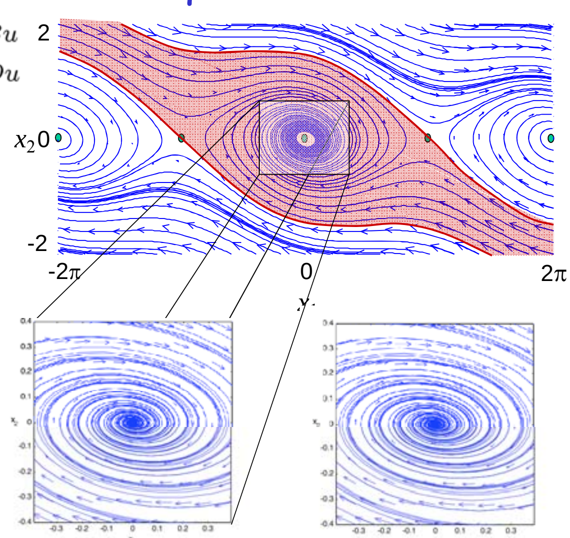
"Linearize" around $x=x_e$

$$f(x_e,u_e)=0 \quad y_e=h(x_e,u_e)$$

$$z=x-x_e \quad v=u-u_e \quad w=y-y_e$$

$$A=\left.\frac{\partial f}{\partial x}\right|_{(x_e,u_e)} \quad B=\left.\frac{\partial f}{\partial u}\right|_{(x_e,u_e)}$$

$$C=\left.\frac{\partial h}{\partial x}\right|_{(x_e,u_e)} \quad D=\left.\frac{\partial h}{\partial u}\right|_{(x_e,u_e)}$$

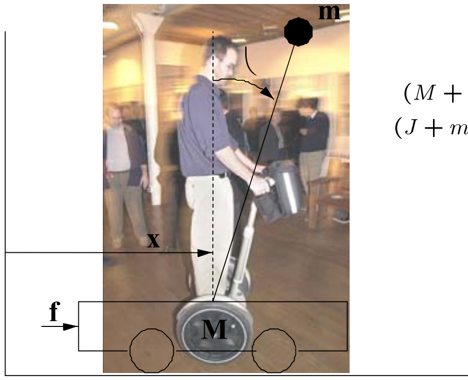


Full nonlinear model Linear model (honest!)

Remarks

- In examples, this is often equivalent to small angle approximations, etc
- Only works *near* equilibrium point

Example: Inverted Pendulum on a Cart



$$(M+m)\ddot{x} + ml\cos\theta\ddot{\theta} = -b\dot{x} + ml\sin\theta\dot{\theta}^2 + f$$
$$(J+ml^2)\ddot{\theta} + ml\cos\theta\ddot{x} = -mgl\sin\theta$$

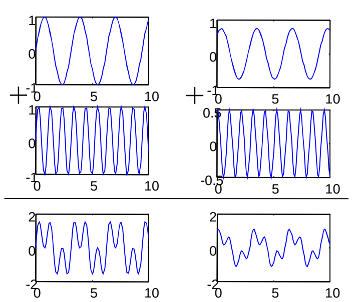
- **State:** $x,\ \theta,\ \dot{x},\ \dot{\theta}$
- **Input:** ***u* = *F***
- **Output:** ***y* = *x***
- **Linearize according to previous formula around θ = 0**

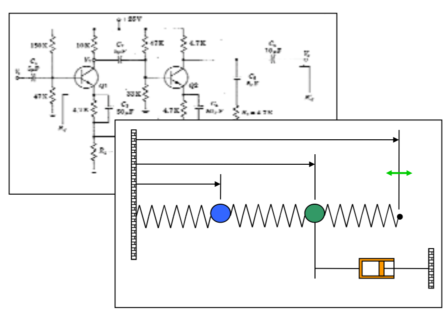
$$\frac{d}{dt}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2gl^2}{J(M+m)+Mml^2} & \frac{-(J+ml^2)b}{J(M+m)+Mml^2} & 0 \\ 0 & \frac{mgl(M+m)}{J(M+m)+Mml^2} & \frac{-mlb}{J(M+m)+Mml^2} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \frac{J+ml^2}{J(M+m)+Mml^2} \\ \frac{ml}{J(M+m)+Mml^2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$$

Summary: Linear Systems

u →
$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \\ x(0) &= 0 \end{aligned}$$
→ y





$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

Properties of linear systems

- Linearity with respect to initial condition and inputs
- Stability characterized by eigenvalues
- Many applications and tools available
- Provide local description for nonlinear systems