CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 110a

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Lyapunov Stability Analysis

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This lecture provides an overview of Lyapunov stability for time-invariant systems. We present a survey of the main results that we shall need in the sequel; proofs are omitted, but can be found in standard texts such as Vidyasagar [2] or Khalil [1].

Reading:

- Åström and Murray, Section 4.4

1 Basic definitions

Consider a closed loop dynamical system of the form

$$\dot{x} = F(x) \qquad x \in \mathbb{R}^n \tag{1}$$

with equilibrium point $x_e \in \mathbb{R}^n$. We recall the following definition from Monday's lecture:

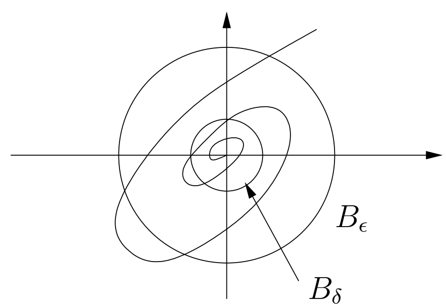
Definition 1. An equlibrium point $x_e = 0$ is *locally asymptotically stable* if

1. $x_e = 0$ is *stable in the sense of Lyapunov*: for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|x(0) - x_e\| < \delta \quad \Longrightarrow \quad \|x(t) - x_e\| < \epsilon \quad \text{for all } t > 0.$$

2. $x_e = 0$ is *locally attractive*: there exists $\delta > 0$ such that

$$\|x(t_0)\| < \delta \quad \Longrightarrow \quad \lim_{t\to\infty} x(t) = 0$$



2 Lyapunov Stability

Definition 2. A continuous function $V : \mathbb{R}^n \to \mathbb{R}$ is *(locally) positive definite* if for some $r > 0$

$$V(0) = 0 \quad \text{and} \quad V(x) > 0 \quad \text{for all } x \neq 0 \text{ and } \|x\| < r.$$

V is *(locally) positive semi-definite* if for some $r > 0$

$$V(0) = 0 \quad \text{and} \quad V(x) \geq 0 \quad \text{for all } \|x\| < r.$$

V is globally positive (semi-) definite if these statements are true for all $x \in \mathbb{R}$.

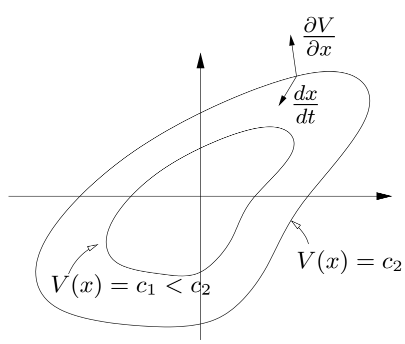
Remarks:

1. To see the difference between positive definite and positive semi-definite, suppose that $x \in \mathbb{R}^2$ and let
$$V_1(x) = x_1^2 \qquad\qquad V_2(x) = x_1^2 + x_2^2.$$
Both V_1 and V_2 are always non-negative. However, it is possible for V_1 to be zero even if $x \neq 0$. Specifically, if we set $x = (0, c)$ where $c \in \mathbb{R}$ is any non-zero number, then $V_1(x) = 0$. On the other hand, $V_2(x) = 0$ if and only if $x = (0, 0)$. Thus $V_1(x)$ is positive semi-definite and $V_2(x)$ is positive definite.

Theorem 1. *Let V be a non-negative function on $\mathbb{R}^n$ and let $\dot{V}$ represent the time derivative of V along trajectories of the system dynamics* (1)*:*
$$\dot{V} = \frac{\partial V}{\partial x}\frac{dx}{dt} = \frac{\partial V}{\partial x}F(x).$$
Let $B_r = B_r(0)$ be a ball of radius r around the origin. If there exists $r > 0$ such that V is positive definite and $\dot{V}$ is negative semi-definite for all $x \in B_r$, then $x = 0$ is locally stable in the sense of Lyapunov. If V is positive definite and $\dot{V}$ is negative definite in B_r, then $x = 0$ is locally asymptotically stable.

Remarks

1. A function V satisfying the conditions of the theorem is called a *Lyapunov function*.
2. $V(x)$ is an "energy like" function that bounds the size of x.



Example 1.

$$\dot{x}_1 = -x_1 - x_2 \qquad V(x) = x_1^2 + x_2^2 > 0 \quad \forall x \neq 0$$
$$\dot{x}_2 = -x_2 \qquad \dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2$$
$$= -2x_1^2 - 2x_1x_2 - 2x_2^2$$
$$= -(x_1 + x_2)^2 - x_1^2 - x_2^2 < 0 \quad \forall x \neq 0$$

$\Longrightarrow$ *globally* asymptotically stable.

Remarks:

1. Q: How do we *find* a Lyapunov function?
   A (pre 2000): guess
   A (post 2000): SOSTOOLS (MATLAB toolbox), for polynomial systems with known coefficients

2. Converse theorem: if $x_e = 0$ is asymptotically stable then there exists V positive definite and $\dot{V}$ negative definite *implies* if you look hard, you can find a Lyapunov function

3. Often the case the $V(x) = x^T P x$ works, where where $P \in \mathbb{R}^{n \times x}$ is a symmetric matrix ($P = P^T$). The condition that V is positive definite is equivalent to the condition that P is a *positive definite* matrix:
$$x^T P x > 0 \qquad \text{for all } x \neq 0,$$
   which we write as $P > 0$. It can be shown that if P is symmetric and positive definite then all of its eigenvalues are real and positive. The level sets of $x^T P x$ are ellipsoids (picture).

Example 2 (Scalar nonlinear system)**.** Consider the scalar nonlinear system

$$\frac{dx}{dt} = \frac{2}{1+x} - x.$$

This system has equilibrium points at $x = 1$ and $x = -2$. We consider the equilibrium point at $x = 1$ and rewrite the dynamics using $z = x - 1$:

$$\frac{dz}{dt} = \frac{2}{2+z} - z - 1,$$

which has an equilibrium point at $z = 0$. Now consider the candidate Lyapunov function

$$V(z) = \frac{1}{2} z^2,$$

which is globally positive definite. The derivative of V along trajectories of the system is given by

$$\dot{V}(z) = z\dot{z} = \frac{2z}{2+z} - z^2 - z.$$

If we restrict our analysis to an interval B_r, where $r < 2$, then $2 + z > 0$ and we can multiply through by $2 + z$ to obtain

$$2z - (z^2 + z)(2 + z) = -z^3 - 3z^2 = -z^2(z + 3) < 0, \qquad z \in B_r,\ r < 2.$$

It follows that $\dot{V}(z) < 0$ for all $z \in B_r$, $z \neq 0$, and hence the equilibrium point $x_e = 1$ is locally asymptotically stable.

Example 3. Consider a damped spring mass system with dynamics

$$m\ddot{q} + c\dot{q} + kq = 0.$$

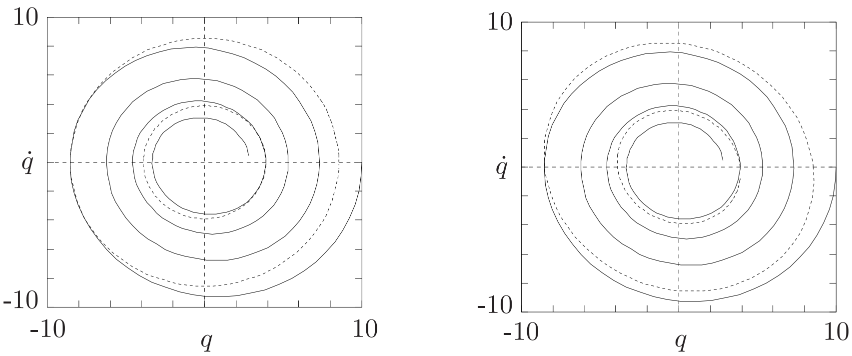
A natural candidate for a Lyapunov function is the total energy of the system, given by

$$V = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2.$$

The derivative of this function along trajectories of the system is

$$\dot{V} = m\dot{q}\ddot{q} + kq\dot{q} = -c\dot{q}^2.$$

This function is only negative semi-definite and hence we cannot conclude asymptotic stability.



To fix this problem, we skew the level sets slightly, so that the flow of the system crosses the level surfaces transversely. Define

$$V(x,t) = \tfrac{1}{2}\begin{bmatrix} q \\ \dot{q} \end{bmatrix}^T \begin{bmatrix} k & \epsilon m \\ \epsilon m & m \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \tfrac{1}{2}\dot{q}m\dot{q} + \tfrac{1}{2}qkq + \epsilon\dot{q}mq,$$

where ϵ is a small positive constant such that V is still positive definite. The derivative of the Lyapunov function becomes

$$\begin{aligned} \dot{V} &= \dot{q}m\ddot{q} + qk\dot{q} + \epsilon m\dot{q}^2 + \epsilon qm\ddot{q} \\ &= (-c + \epsilon m)\dot{q}^2 + \epsilon(-kq^2 - cq\dot{q}) = -\begin{bmatrix} q \\ \dot{q} \end{bmatrix}^T \begin{bmatrix} \epsilon k & \frac{1}{2}\epsilon c \\ \frac{1}{2}\epsilon c & c - \epsilon m \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix}. \end{aligned}$$

The function $\dot{V}$ can be made negative definite for ϵ chosen sufficiently small (exercise) and hence we can conclude *exponential* stability.

Remarks

1. As the previous example shows, a Lyapunov function need not be unique and different Lyapunov functions can give stronger stability results.

2. Lyapunov functions can also be used to prove that a system is unstable: search for V positive definite with $\dot{V}$ positive definite.

3 Lyapunov Functions for Linear Systems

Consider a linear system of the form

$$\dot{x} = Ax.$$

Search for a quadratic Lyapunov function

$$V(x) = x^T P x$$

Compute the derivative

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} = x^T(A^T P + PA)x.$$

The requirement that V is positive definite is equivalent to $P > 0$ and the requirement that $\dot{V}$ is negative definite becomes a condition that

$$Q = A^T P + PA < 0 \tag{2}$$

(as a matrix).

Trick: equation (2) is *linear* in P. So we can *choose* $Q < 0$ and the *solve* for P.

Example 4. Consider the linear system

$$\begin{aligned} \frac{dx_1}{dt} &= -ax_1 \\ \frac{dx_2}{dt} &= -bx_1 - cx_2. \end{aligned}$$

with $a, b, c > 0$, for which we have

$$A = \begin{bmatrix} -a & 0 \\ -b & -c \end{bmatrix} \qquad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}.$$

We choose $Q = -I \in \mathbb{R}^{2\times 2}$ and the corresponding Lyapunov equation is

$$\begin{bmatrix} -a & -b \\ 0 & -c \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} -a & 0 \\ -b & -c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and solving for the elements of P yields

$$P = \begin{bmatrix} \frac{b^2+ac+c^2}{2a^2c+2ac^2} & \frac{-b}{2c(a+c)} \\ \frac{-b}{2c(a+c)} & \frac{1}{2} \end{bmatrix}$$

or

$$V(x) = \frac{b^2+ac+c^2}{2a^2c+2ac^2} x_1^2 - \frac{b}{c(a+c)} x_1 x_2 + \frac{1}{2} x_2^2.$$

It is easy to verify that $P > 0$ (check its eigenvalues) and by construction $\dot{P} = -I < 0$. Hence the system is asymptotically stable.

4 Krasolvskii-Lasalle Invariance Principle (optional)

The Krasolvskii-Lasalle invariance principle provides a way to prove asymptotic stability when $\dot{V}$ is negative semi-definite (which is usually much easier to find).

Denote the solution trajectories of the time-invariant system

$$\frac{dx}{dt} = F(x) \tag{3}$$

as $x(t; x_0, t_0)$, which is the solution of equation (3) at time t starting from x_0 at t_0. We write $x(\cdot; x_0, t_0)$ for the set of all points lying along the trajectory.

Definition 3 (ω limit set)**.** The ω *limit set* of a trajectory $x(\cdot; x_0, t_0)$ is the set of all points $z \in \mathbb{R}^n$ such that there exists a strictly increasing sequence of times t_n such that

$$s(t_n; x_0, t_0) \to z$$

as $n \to \infty$.

Definition 4 (Invariant set)**.** The set $M \subset \mathbb{R}^n$ is said to be an *invariant set* if for all $y \in M$ and $t_0 \geq 0$, we have

$$x(t; y, t_0) \in M \quad \text{for all } t \geq t_0.$$

Theorem 2 (Krasovskii-Lasalle principle)**.** *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally positive definite function such that on the compact set $\Omega_r = \{x \in \mathbb{R}^n : V(x) \leq r\}$ we have $\dot{V}(x) \leq 0$. Define*

$$S = \{x \in \Omega_r : \dot{V}(x) = 0\}.$$

As $t \to \infty$, the trajectory tends to the largest invariant set inside S; i.e., its ω limit set is contained inside the largest invariant set in S. In particular, if S contains no invariant sets other than $x = 0$, then 0 is asymptotically stable.

Example 5 (Damped spring mass system)**.** Consider a damped spring mass system with dynamics

$$m\ddot{q} + c\dot{q} + kq = 0.$$

A natural candidate for a Lyapunov function is the total energy of the system, given by

$$V = \frac{1}{2}m\dot{q}^2 + \frac{1}{2}kq^2.$$

The derivative of this function along trajectories of the system is

$$\dot{V} = m\dot{q}\ddot{q} + kq\dot{q} = -c\dot{q}.$$

This function is only negative semi-definite and hence we cannot conclude asymptotic stability using Theorem 1. However, note that $\dot{V} = 0$ implies that $\dot{q} = 0$. If we define

$$S = \{(q, \dot{q}) : \dot{q} = 0\}$$

then we can compute the largest invariant set inside S. For this set, we must have $\dot{q}(t) = 0$ for all t and hence $\ddot{q}(t) = 0$ as well.

Using the dynamics of the system, we see that if $\dot{q}(t) = 0$ and $\ddot{q}(t) = 0$ then $\dot{q}(t) = 0$ as well. hence the largest invariant set inside S is $(q, \dot{q}) = 0$ and we can use the Krasovskii-Lasalle principle to conclude that the origin is asymptotically stable. Note that we have not made use of Ω_r in this argument; for this example we have $\dot{V}(x) \leq 0$ for any state and hence we can choose r arbitrarily large.

References

[1] H. K. Khalil. *Nonlinear Systems*. Macmillan Publishing Company, 1992.

[2] M. Vidyasagar. *Nonlinear Systems Analysis*. Prentice-Hall, second edition edition, 1993.