



CDS 101/110a: Lecture 2.1 Dynamic Behavior



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Goals:

- Learn to use phase portraits to visualize behavior of dynamical systems
- Understand different types of stability for an equilibrium point
- Know the difference between local/global stability and related concepts

Reading:

- Åström and Murray, *Feedback Systems*, Chapter 4 [90 minutes]
- Advanced: S. H. Strogatz, *Nonlinear Dynamics and Chaos*, Chapter 6

Modeling Terminology

$$\dot{x} = f(x, u)$$

$$y = h(x, u)$$

State captures effects of the past

- independent physical quantities that determines future evolution (absent external excitation)
- Not unique (e.g. units)

Inputs describe external excitation

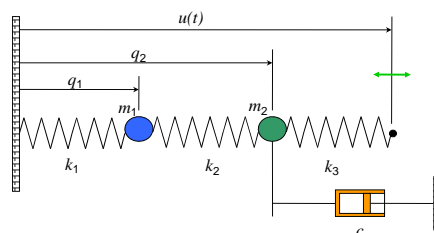
- Inputs are *extrinsic* to the system dynamics (externally specified)
- Disturbances & control inputs

Dynamics describes state evolution

- update rule for system state
- function of current state and any external inputs

Outputs describe measured quantities

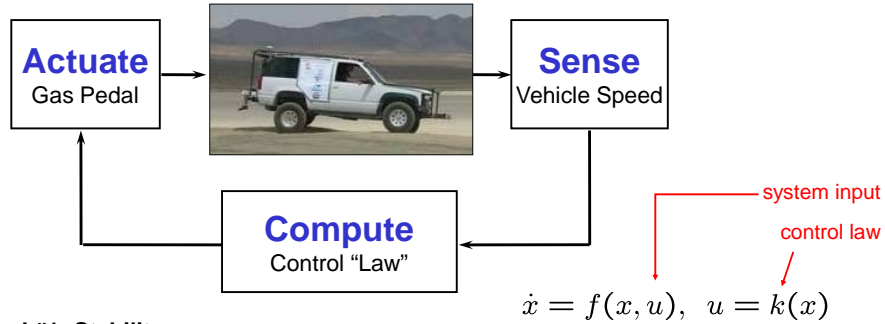
- Outputs are function of state and inputs \Rightarrow not independent variables
- Outputs are often *subset* of state



Example: spring mass system

- State: position and velocity of each mass: $q_1, q_2, \dot{q}_1, \dot{q}_2$
- Input: position of spring at right end of chain: $u(t)$
- Dynamics: basic mechanics
- Output: measured positions of the masses: q_1, q_2

Dynamic Behavior (and Stability)

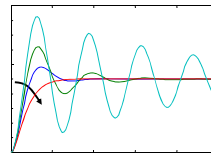


Goal #1: Stability

- Check if closed loop response is stable $\Rightarrow \dot{x} = f(x, k(x)) = F(x)$

Goal #2: Performance

- Look at how the closed loop system behaves, in a dynamic context



Response depends on choice of control (all are stable)

Goal #3: Robustness (later)

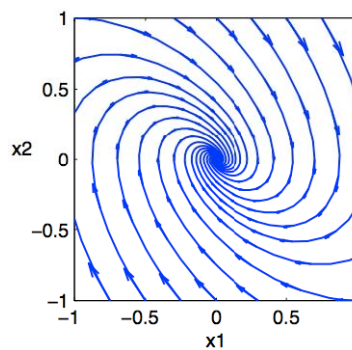
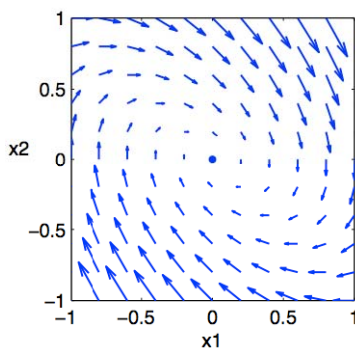
Phase Portraits (2D systems only)

Phase plane plots show 2D dynamics as *vector fields* & *stream functions*

- $\dot{x} = f(x, u(x)) = F(x)$
- Plot $F(x)$ as a vector on the plane; stream lines follow the flow of the arrows

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 - x_2 \end{bmatrix}$$

```
phaseplot('dosc', ...
[-1 1 10], [-1 1 10], 0.1, ...
boxgrid([-1 1 10], [-1 1 10]));
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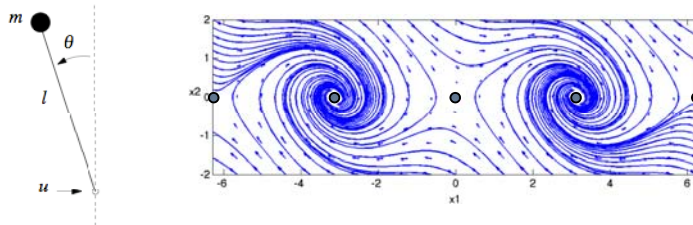


Equilibrium Points

Equilibrium points represent stationary conditions for the dynamics

- The *equilibria* of the system $\dot{x} = F(x)$ are the points x_e such that $f(x_e) = 0$.

$$\frac{dx}{dt} = \begin{bmatrix} x_2 \\ \sin x_1 - \gamma x_2 \end{bmatrix} \Rightarrow x_e = \begin{bmatrix} \pm n\pi \\ 0 \end{bmatrix}$$

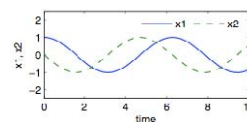
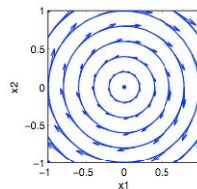


Stability of Equilibrium Points

An equilibrium point is:

Stable if initial conditions that start near the equilibrium point, stay near

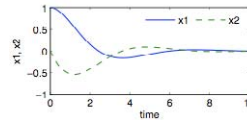
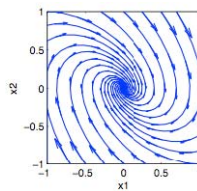
- Also called “stable in the sense of Lyapunov”



$$\|x(0) - x_e\| < \delta \Rightarrow \|x(t) - x_e\| < \epsilon$$

Asymptotically stable if all nearby initial conditions converge to the equilibrium point

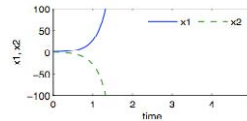
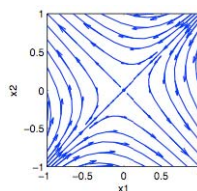
- Stable + converging



$$\lim_{t \rightarrow \infty} x(t) = x_e \quad \forall \|x(0) - x_e\| < \epsilon$$

Unstable if some initial conditions diverge from the equilibrium point

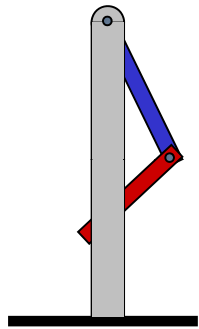
- May still be some initial conditions that converge



Example #1: Double Inverted Pendulum

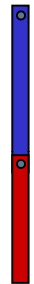
Two series coupled pendula

- States: pendulum angles (2), velocities (2)
- Dynamics: $F = ma$ (balance of forces)
- Dynamics are very nonlinear



Eq #1

Eq #2



Eq #3

Eq #4

Stability of equilibria

- Eq #1 is stable
- Eq #3 is unstable
- Eq #2 and #4 are unstable, but with some stable "modes"

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Stability of Linear Systems

Linear dynamical system with state $x \in \mathbb{R}^n$:

$$\frac{dx}{dt} = Ax \quad x(0) = x_0,$$

Stability determined by the eigenvalues $\lambda(A) = \{s \in \mathbb{C} : \det(sI - A) = 0\}$

- Simplest case: diagonal A matrix (all eigenvalues are real)

$$\frac{dx}{dt} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} x$$

$$\dot{x}_i = \lambda_i x_i$$

$$x_i(t) = e^{\lambda_i t} x_i(0)$$

- System is asy stable if $\lambda_i < 0$

- Block diagonal case (complex eigenvalues)

$$\frac{dx}{dt} = \begin{bmatrix} \sigma_1 & \omega_1 & 0 & 0 \\ -\omega_1 & \sigma_1 & 0 & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \sigma_m & \omega_m \\ 0 & 0 & -\omega_m & \sigma_m \end{bmatrix} x$$

$$x_{2j-1}(t) = e^{\sigma_j t} (x_i(0) \cos \omega_j t + x_{i+1}(0) \sin \omega_j t)$$

$$x_{2j}(t) = e^{\sigma_j t} (x_i(0) \sin \omega_j t - x_{i+1}(0) \cos \omega_j t)$$

- System is asy stable if $\text{Re } \lambda_i = \sigma_i < 0$

- **Theorem:** linear system is asymptotically stable if and only if $\text{Re } \lambda_i < 0 \quad \forall \lambda_i \in \lambda(A)$

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Local Stability of Nonlinear Systems

Asymptotic stability of the linearization implies *local* asymptotic stability of equilibrium point

- Linearization around equilibrium point captures “tangent” dynamics

$$\dot{x} = F(x_e) + \frac{\partial F}{\partial x} \Big|_{x_e} (x - x_e) + \text{higher order terms} \xrightarrow{\text{approx}} \begin{matrix} \dot{z} = A z \\ z = x - x_e \end{matrix}$$

- If linearization is *unstable*, can conclude that nonlinear system is locally unstable
- If linearization is *stable* but not *asymptotically stable*, can't conclude anything about nonlinear system:

$$\dot{x} = \pm x^3 \xrightarrow{\text{linearize}} \dot{x} = 0$$

- linearization is stable (but not asy stable)
- nonlinear system can be asy stable or unstable

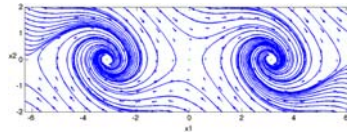
Local approximation particularly appropriate for control systems design

- Control often used to *ensure* system stays near desired equilibrium point
- If dynamics are well-approximated by linearization near equilibrium point, can use this to design the controller that keeps you there (!)

Example: Stability Analysis of Inverted Pendulum

System dynamics

$$\frac{dx}{dt} = \begin{bmatrix} x_2 \\ \sin x_1 - \gamma x_2 \end{bmatrix},$$

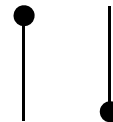


Upward equilibrium:

- $\theta = x_1 \ll 1 \implies \sin x_1 \approx x_1$

$$\frac{dx}{dt} = \begin{bmatrix} x_2 \\ x_1 - \gamma x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -\gamma \end{bmatrix} x$$

- Eigenvalues: $-\frac{1}{2}\gamma \pm \frac{1}{2}\sqrt{4 + \gamma^2}$



Downward equilibrium:

- Linearize around $x_1 = \pi + z_1$: $\sin(\pi + z_1) = -\sin z_1 \approx -z_1$

$$\begin{matrix} z_1 = x_1 - \pi \\ z_2 = x_2 \end{matrix} \implies \frac{dz}{dt} = \begin{bmatrix} z_2 \\ -z_1 - \gamma z_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\gamma \end{bmatrix} z$$

- Eigenvalues: $-\frac{1}{2}\gamma \pm \frac{1}{2}\sqrt{-4 + \gamma^2}$

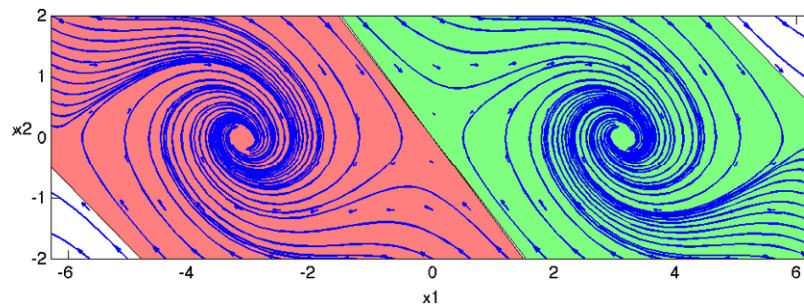
Local versus Global Behavior

Stability is a *local* concept

- Equilibrium points define the local behavior of the dynamical system
- Single dynamical system can have stable *and* unstable equilibrium points

Region of attraction

- Set of initial conditions that converge to a given equilibrium point



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Example #2: Predator Prey (ODE version)

Continuous time (ODE) version of predator prey dynamics:

$$\begin{aligned} \frac{dH}{dt} &= rH \left(1 - \frac{H}{k}\right) - \frac{aHL}{c+H} & H \geq 0 \\ \frac{dL}{dt} &= b \frac{aHL}{c+H} - dL & L \geq 0. \end{aligned}$$

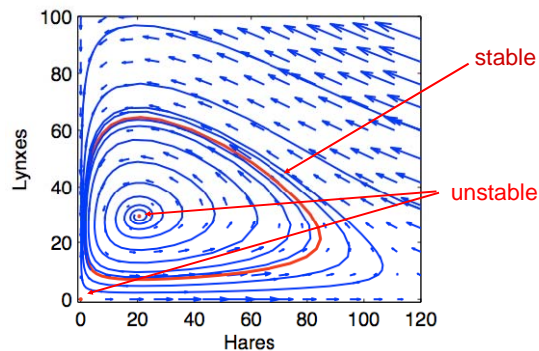
- Continuous time (ODE) model
- MATLAB: predprey.m (from web page)

Equilibrium points (3)

- $\sim(20.5, 29.5)$: unstable
- $(0, 0)$: unstable
- $(125, 0)$: unstable

Limit cycle

- Population of each species oscillates over time
- Limit cycle is stable (nearby solutions converge to limit cycle)
- This is a *global* feature of the dynamics (not local to an equilibrium point)



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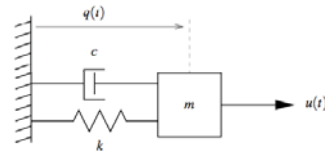
Reasoning about Stability using Lyapunov Functions

Basic idea: capture the behavior of a system by tracking “energy” in system

- Find a single function that captures distance of system from equilibrium
- Try to reason about the long term behavior of *all* solutions

Example: spring mass system

- Can we show that all solutions return to rest w/out explicitly solving ODE?
- Idea: look at how energy evolves in time



$$m\ddot{q} + c\dot{q} + kq = 0$$

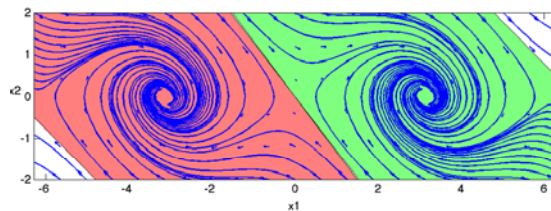
- Start by writing equations in state space form
- Compute energy and its derivative

$$\frac{dx}{dt} = \begin{bmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{c}{m}x_2 \end{bmatrix} \quad \begin{matrix} x_1 = q \\ x_2 = \dot{q} \end{matrix}$$

$$\begin{aligned} V(x) &= \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2 & \frac{dV}{dt} &= kx_1x_1 + mx_2x_2 \\ & & &= kx_1x_2 + mx_2\left(-\frac{c}{m}x_2 - \frac{k}{m}x_1\right) = -cx_2^2 \end{aligned}$$

- Energy is positive $\Rightarrow x_2$ must eventually go to zero
- If x_2 goes to zero, can show that x_1 must also approach zero (Krasovskii-Lasalle)

Summary: Stability and Performance



Key topics for this lecture

- Stability of equilibrium points
- Eigenvalues determine stability for linear systems
- Local versus global behavior

