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J_2 DYNAMICS AND FORMATION FLIGHT

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Abstract

We study the dynamics of the relative motion of satellites in the gravitational field of the Earth, including the effects of the bulge of the Earth (the J_2 effect). Using Routh reduction and dynamical systems ideas, a method is found that locates orbits such that a cluster of satellites remains close with very little dispersing, even with no controls.

1 Introduction

There has been considerable interest in distributing the functions of a single large satellite among several small cooperative units. Many potential applications of this enabling technology exist, one of which is to improve the performance of Earth observation. A cluster of satellites will be able to synthesize a much larger aperture than can be achieved with a single platform, thus providing significant increases in image resolution through interferometry.

Advantages of Clusters. In addition to the main advantage of having a long baseline, there are other advantages of using a cluster of satellites, including the following:

1. Individual vehicles are inexpensive, which allows for redundancy of critical components.
2. Since the number of vehicles in the cluster is arbitrary, the cluster can be expanded if new demands arise.
3. Because the cluster can be reconfigured, it is flexible and can be changed to perform new mission objectives.
4. The cluster can be updated to include new sensors by adding only a few satellites.
5. Should one satellite in the cluster malfunction, the mission may continue with the remaining satellites, and the cluster can be repaired either through replacement of the malfunctioning vehicle or reconfiguration of the cluster.

Recent Advances in Formation Flight. Sedwick, Kong and Miller[1] used the Clohessy-Wilshire equations as their starting model to find relative orbits about a reference spacecraft. A circular reference orbit and a spherical Earth (without J_2 effect) was assumed in their study. The equations of motion of the other spacecraft were linearized relative to the rotating frame of the reference spacecraft. They used these linear equations of motion to establish a large family of relative orbits. Their estimations showed that only a small amount of fuel (roughly 20 meters per year per spacecraft) was required if only the differential perturbations that tend to affect the size and shape of the cluster were addressed. In the Earth orbit at about 800 km altitude, the J_2 effect is much larger in comparison with other perturbations such as atmospheric drag, solar radiation pressure and electro-magnetic effects.

Responding to the need to take into consideration both the nonlinearity and the J_2 effect right from the start, Schaub and Afriend[2] built on the work of Brouwer[3] and found J_2 invariant relative orbits. Working with mean orbit elements, the secular drift of the longitude of the ascending node and the sum of the argument of perigee and mean anomaly were set equal between two neighboring orbits. By having both orbits drift at equal rates on the average, they would not pull apart over time due to the J_2 influence. Two first order conditions were established between the differences in momentum elements (semi-major axis, eccentricity and inclination angle) that guarantee that the drift rates of two neighboring orbits were equal on the average. Differences in the longitude of the ascending node, argument of perigee and initial mean anomaly could be set at will, as long as they were setup in mean element space.

Inspired by our joint work with the Jet Propulsion Laboratory in applying dynamical systems theory to space mission design near the libration points (see Koon, Lo, Marsden and Ross[4]), we have developed similar dynamical systems techniques appropriate to the near Earth case and found a family of candidate reference orbits whose nearby orbits support formation flight. Using Routh reduction and Poincaré section techniques appropriate for the J_2 dynamics, we have developed a procedure for locating orbits such that the cluster of satellites remains close for many years, with very little dispersing, even with no controls. Rather than using orbital elements, our analysis is done directly in physical space which makes the connection with physical requirements more direct.

This methodology of finding dynamically-favorable orbits, if coupled with control and optimal control, may provide an effective way to deal with maintenance and reconfiguration of formation flight of near Earth satellites, as well as providing the mission designer with a complete picture of fuel-efficient formations.

2 The Reduced Equations

In this section, we will use the Routh reduction technique to rewrite the equations of motion of the full system in a simpler form. This procedure will enable us to study first the reduced dynamics in the meridian plane of the satellite before dealing with the dynamics in the longitudinal direction.

Recall that in spherical coordinates (ρ, ϕ, θ), the potential energy including the J_2 effect is given by

$$U = -\frac{\mu}{\rho} + \frac{\mu R_e^2 J_2}{\rho^3}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right).$$

where μ is the gravitational constant of the Earth ($\mu = GM_e = 3.986005\times 10^{14} m^3/s^2$), R_e is the radius of the Earth ($R_e = 6378140m$) and J_2 is the second zonal harmonic coefficient due to the oblateness of the Earth ($J_2 = 0.00108263$).

Following Broucke[5], we use the z-axis symmetry of the J_2 problem where the longitude variable ϕ is ignorable and the z component of the angular momentum is conserved to reduce the equations of motion into two second order equations with a Routhian function

$$R = \frac{1}{2}(\dot\rho^2 + \rho^2\dot\theta^2) - \frac{H_z^2}{2\rho^2\sin^2\theta} - U$$

where H_z is the z component of the angular momentum.

In the rectangular coordinates (r, z) of the corotating meridian plane of the satellite, the Routhian function becomes

$$R = \frac{1}{2}(\dot r^2 + \dot z^2) - \frac{H_z^2}{2r^2} - U(r,z).$$

where $\rho^2 = r^2 + z^2$ and $\cos\theta = z/\rho$. The reduced equations are then given by

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot r}\right) = \frac{\partial R}{\partial r}, \qquad \frac{d}{dt}\left(\frac{\partial R}{\partial \dot z}\right) = \frac{\partial R}{\partial z}.$$

Equivalently,

$$\begin{aligned}
\ddot r &= H_z^2\frac{1}{r^3} - \mu\frac{r}{(r^2+z^2)^{3/2}} \\
&\quad - \frac{3\mu R_e^2 J_2}{2}\frac{r}{(r^2+z^2)^{5/2}} \\
&\quad + \frac{15\mu R_e^2 J_2}{2}\frac{rz^2}{(r^2+z^2)^{7/2}}; \\
\ddot z &= -\mu\frac{z}{(r^2+z^2)^{3/2}} - \frac{3\mu R_e^2 J_2}{2}\frac{z}{(r^2+z^2)^{5/2}} \\
&\quad + \frac{3\mu R_e^2 J_2}{2}\frac{(3z^2-2r^2)z}{(r^2+z^2)^{7/2}}.
\end{aligned}$$

Hence, after a Routh reduction, the equations of motion of the full system has been rewritten in a simpler form which enables one to first study the reduced dynamics in the meridian plane (r, z) before dealing with the dynamics in the longitudinal variable ϕ. The general form of the preceding equations is

$$\ddot r = f(r,z), \qquad \ddot z = g(r,z), \qquad \dot\phi = \frac{H_z}{r^2}.$$

Notice also that the energy E given by

$$E = \frac{1}{2}(\dot r^2 + \dot z^2) + \frac{H_z^2}{2r^2} + U(r,z) \tag{1}$$

is the other integral of motion besides H_z.

3 The Pseudo-circular Orbit and A Cluster of Micro-satellites

After performing Routh reduction, we are ready to use the method of Poincaré section in finding the initial conditions for orbits that are dynamically favorable to formation flight.

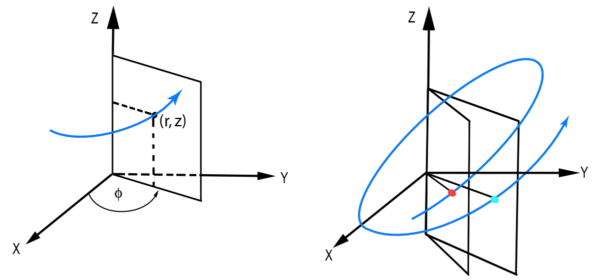

Figure 1: The plane $z = 0$ is the plane of the Earth's equator. Poincaré section is made by plotting a point $(r, \dot{r})$ whenever the satellite crosses the equator from the south to the north.

Poincaré Section. Since the energy E is conserved (in the meridian variables (r, z)), the constant energy surface for the reduced system is three dimensional and the hyperplane $z = 0$ can be used as the transversal plane to obtain the two dimensional Poincaré section. See Figure 1 and Figure 2. Notice that the plane $z = 0$ is the plane of the Earth's equator. Roughly speaking, the Poincaré section is made by plotting a point $(r, \dot{r})$ whenever the satellite crosses the equator from the south to the north.

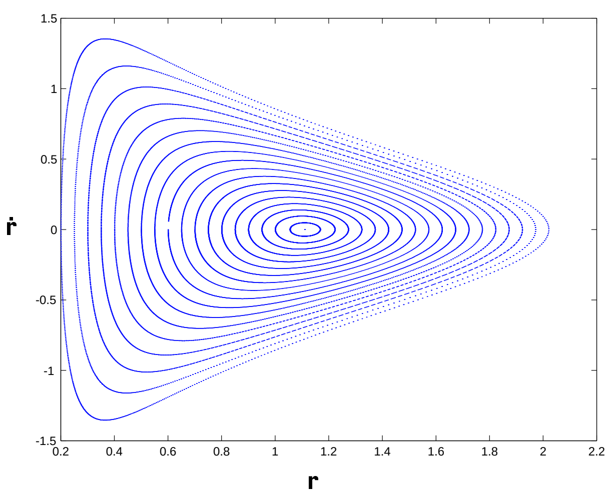

Figure 2: $E = -0.45, H_z^2 = 0.3$. Poincaré section of $(r, \dot{r})$ at $z = 0$. Units of time and length have been chosen to make the radius and the gravitational constant of the Earth equal to 1.

As the z component (H_z) of angular momentum (which is related to the inclination) is varied from its maximum value $(-1/2E)$ for to its minimum value (zero) from equator to polar, a number of interesting bifurcations take place, especially around the critical inclination (see Figure 3). However, since our main interest in this study is on formation flight, we will refer the readers who are interested in this bifurcation analysis to Broucke[5]. Also, in this paper, we have concentrated our attention on the general case which is neither polar nor near critical inclination. But we intend to study these interesting cases in our next paper.

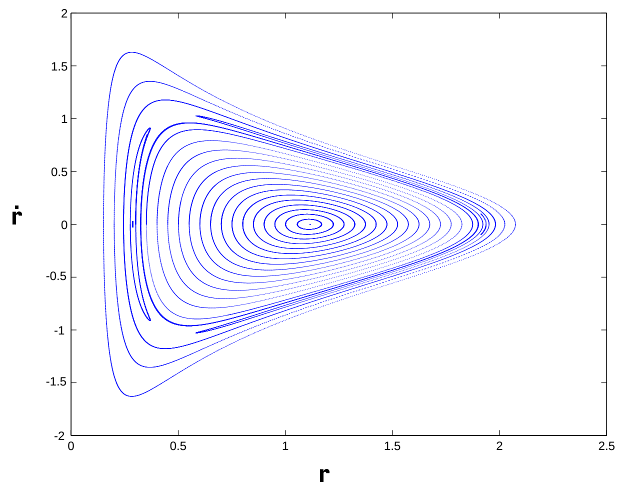

Figure 3: $E = -0.45, H_z^2 = 0.1$. Poincaré section of $(r, \dot{r})$ at $z = 0$. Figure shows the bifurcation near the critical inclination where the Poincaré map has 3 stable and 2 unstable fixed points.

Notice that for any fixed values of E and H_z, each point $(r, \dot{r})$ of the Poincaré section gives the **initial conditions** $(r, z, \phi, \dot{r}, \dot{z}, \dot{\phi})$ for an orbit of the full system. This is because $z = 0$ and $\dot{z} = \dot{z}(r, \dot{r}, 0, E, H_z)$ (where $\dot{z} > 0$) and $\dot{\phi} = H_z/r^2$ can be computed from the fixed energy E and the fixed z-component of angular momentum H_z once $(r, \dot{r})$ are known. Also, since ϕ is ignorable, it can be chosen arbitrary. For convenience sake, we can set $\phi = 0$ at $t = 0$. Hence, $(r, \dot{r})$ (or more fully, $(r, 0, \phi, \dot{r}, E, H_z)$) provides all the initial conditions for an orbit of the full system.

The Pseudo-Circular Orbit. By studying this Poincaré section (Figure 2) and looking for the stable fixed point, we can find the *pseudo-circular orbit* (which corresponds to the fixed point in the middle of Figure 2) whose nearby orbits can be used for formation flight. Clearly, this fixed point corresponds to a periodic orbit in the reduced system. But it also gives rise to a trajectory that is also periodic in some sense in the full system: modulo the precession in ϕ in a revolution around the earth, this trajectory repeats itself.

As pointed out in Broucke[5], this pseudo-circular orbit whose mean eccentricity is nearly 0 is the central backbone of a whole set of solutions (the closed curves surrounding the fixed point in Figure 2). The other solutions on the Poincaré section are the quasi-periodic solutions, which are elliptic orbits with precessing perigee locations. The perigee and apogee altitude of these elliptic orbits can be estimated by the two intersections of the invariant curve with the r-axis. These points being at approximately $(a(1-e), 0)$ and $(a(1+e), 0)$ where $(a, 0)$ is the fixed point. Hence, the eccentricity e can also be estimated. Roughly speaking, the set of solutions is parameterized by the eccentricity. Think of each closed curve surrounding the fixed point as a set of solutions which all have the same eccentricity. As one moves out from the fixed point, the eccentricity gets large. Notice that our units of time and length have been chosen in such a way that $R_e = 1$ and $\mu = 1$.

In figure 4 and 5 we show two examples of orbits represented in the rotating meridian plane (r, z). Figure 4 is a pseudo-circular orbit with inclination equal to 58^0. This shows an analogy with the pendulum problem: the motion is merely an up and down libration about the equator. In Figure 5 we show a precessing elliptic orbit. The precession of the perigee and apogee are clearly visible. These figure also show the regions of allowable motion and the zero-velocity curves.

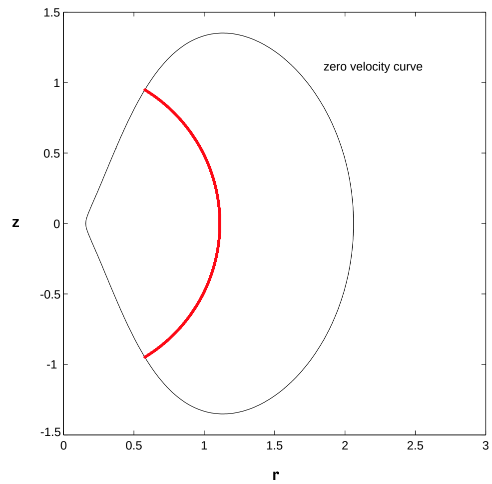

Figure 4: The red curve is the pseudo-circular orbit represented in the rotating meridian plane (r, z). This orbit corresponds to the fixed point in Figure 2. This figure also shows the zero-velocity (black) curve and the region of allowable motion.

Triangular Cluster near the Pseudo-Circular Orbit. By using the fixed point and the points nearby as well as making slight changes in the longitudinal angle ϕ (and possibly in the time t), we can construct different kinds of cluster which will remain together after many years (corresponding to thousands of revolutions around the Earth). For example, if we fix $E = -0.45, H_z^2 = 0.3$, the fixed point for the Poincaré section at $z = 0$ will be $(r_f, 0)$ where $r_f = 1.11133496883$ (about 710 km above the Earth). The following initial conditions will give a triangular cluster (with each side close to 100 meters).

$r - r_f$	$\dot{r}$	ϕ
0.0	0.0	10^{-5}
7×10^{-6}	0.0	0.0
0.0	0.0	-10^{-5}

The evolution of these three satellites in a triangular cluster were plotted in a frame whose origin is at their instantaneous barycenter, with the yz-plane orthogonal to the line of sight, the x-axis pointing towards the center of the Earth, and the y-axis and the z-axis pointing towards the (instantaneous) west and north respectively. Figure 6 shows the trajectories of these three satellites projected onto the yz-plane for 100 revolutions around the Earth (about a week). Figure 7 shows the trajectories of the same satellite cluster in the yz-plane for 5000 revolutions around the Earth (about a year). Notice how small the dispersion is during a year—it measures just a few meters. Recall that the length units have been chosen to make the radius of the Earth (6.4×10^6 meters) equal to 1.

4 A Short Explanation of Why the Method Works So Well

Besides the need to form a triangular cluster orthogonal to the line of sight, the initial conditions of the cluster have been chosen with three main considerations in mind: (i) the three satellites have the same energy E, (ii) they have the same z-component of the angular momentum H_z and (iii) they are near a pseudo-circular orbit whose mean eccentricity e is nearly 0.

Our Poincaré map method essentially ignores short period oscillations (within a single revolution around the Earth) but provides a global view of the

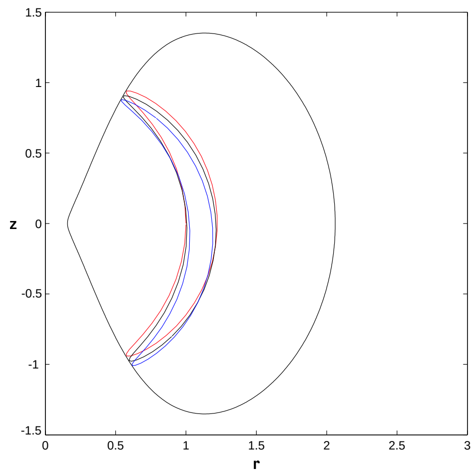

Figure 5: A precessing elliptic orbit represented in the rotating meridian plane (r, z). Shown are three snap shoots at three different instances that are weeks apart.

long period growth and the secure growth caused by the J_2 perturbation. Its success can be explained by the same theory developed by Brouwer[3] and was used in Schaub and Alfriend[2]. Here, we will only sketch the basic approach.

Mean Orbital Elements. In the mean orbital element space, the energy E can be approximated to the first order by

$$E = -\frac{1}{2L^2} + J_2 \frac{1}{4L^6}\left(\frac{L}{G}\right)^3\left(1 - 3\frac{H^2}{G^2}\right). \qquad (2)$$

Here L is the action, G is the angular momentum and H is its z component. They are defined by the following relations:

$$\begin{aligned} L &= \sqrt{a} \\ G &= \sqrt{a(1-e^2)} = L\eta \\ H &= G\cos i \end{aligned}$$

where a is semi-major axis, e is the eccentricity, i is the inclination and $\eta = \sqrt{1-e^2}$.

The condition (ii) means $\delta H = 0$. It can be shown that the conditions (i) and (iii) imply that $\delta L = o(J_2 \delta G)$ and $\delta G = o(\delta\eta)$. Since $\delta\eta$ is really small for any two orbits near a pseudo-circular orbit (in the order of 10^{-11} in our example), the secular drift rate between them are essentially zero. Hence, $\delta L, \delta G$ and δH are nearly zero.

Recall that the mean angle rates $\dot{l}, \dot{g}, \dot{h}$ for the mean anomaly, the argument of perigee and the longitude of ascending node are given by

$$\begin{aligned} \dot{l} &= \frac{1}{L^3} - J_2 \frac{3}{4L^7}\left(\frac{L}{G}\right)^3\left(1 - 3\frac{H^2}{G^2}\right) \\ \dot{g} &= -J_2 \frac{3}{4L^7}\left(\frac{L}{G}\right)^4\left(1 - 5\frac{H^2}{G^2}\right) \\ \dot{h} &= -J_2 \frac{3}{2L^7}\left(\frac{L}{G}\right)^4\left(\frac{H}{G}\right) \end{aligned}$$

A straightforward computation will show that $\delta\dot{l}, \delta\dot{g}, \delta\dot{h}$ are nearly zero. Therefore, the cluster will remain close together with very little dispersing under the natural dynamics.

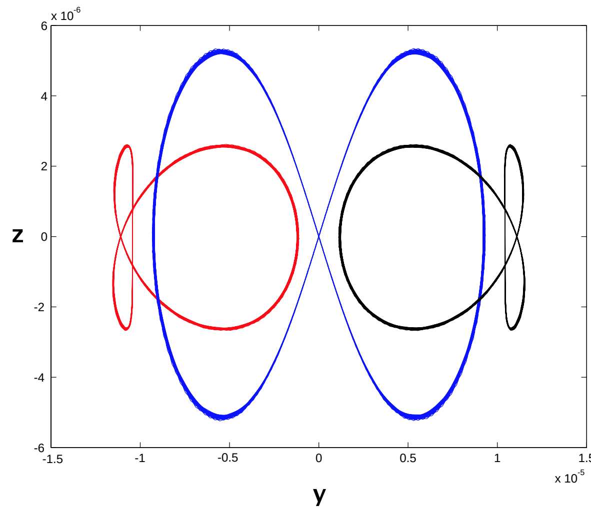

Figure 6: The trajectories of three satellites in the yz-plane for 100 revolutions around the Earth. Unit of length have been chosen to make the radius of the Earth (6.4×10^6 meters) equal to 1.

Slight Modification of Brouwer's Theory. However, for the case of small eccentricity, the perigee argument g and the mean anomaly l are not well-defined. Fortunately, this difficulty is not essential and can be solved by using new orbital elements. In Deprit and Rom[6], the author introduced three new elements F, C and S:

$$\begin{aligned} F &= l + g, \\ C &= e\cos g, \\ S &= e\sin g. \end{aligned}$$

F is nothing but the *mean distance* to the ascending node. C and S are the two components of the eccentricity vector. These three new elements F, C and

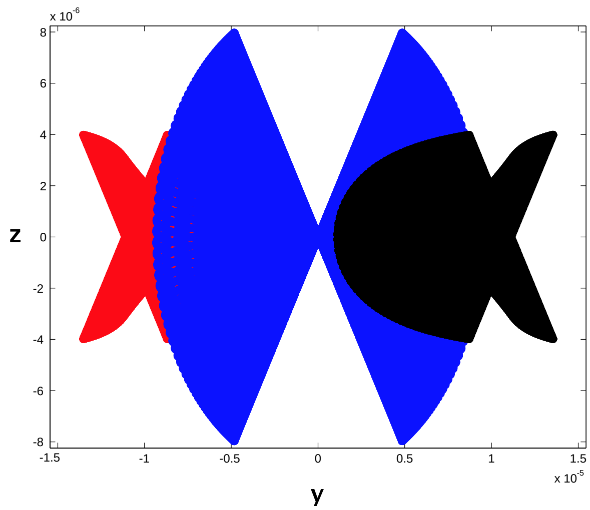

Figure 7: The trajectories of the same three satellites in the yz-plane after 5000 revolutions around the Earth.

S, together with the longitude of ascending node h, the polar component of angular momentum H and the action L, form a complete set of new orbital elements.

For the two most important drifts, namely, the mean distance to the ascending node F and the longitude of ascending node h, it was shown that

$$\dot{F} = \dot{l} + \dot{g},$$

and $\dot{h}$ remains the same. Hence, similar computations will show that the rate of relative drifts for the mean distance $(\dot{\delta F})$ and the longitude of ascending node $(\dot{\delta h})$ among the satellites of the cluster are also nearly zero.

5 Conclusion

Using Routh reduction and Poincaré section techniques appropriate for the J_2 dynamics, we have developed a procedure for locating orbits such that the cluster of satellites remains close for many years, with very little dispersion, even with no controls. This result, if coupled with optimal control techniques, is expected to provide a fuel-efficient way to deal with maintenance and reconfiguration of formation flight.

Merging with Control and Optimal Control. The performance requirements of many formation flight missions, with regard to both formation maintenance and propellant consumption, necessitates the use of low-thrust trajectories and continuous control paradigms. Theoretically, one of the most favored approaches is to use optimal control in generating the low-thrust trajectories. There are a number of numerical difficulties. Previously developed numerical algorithms do not converge for any problem that is relatively sensitive. However, in Milam, Petit and Murray[7] and Petit, Milam and Murray[8] the authors have shown that it is possible to use the *Nonlinear Trajectory Generation* (NTG) software package for maintaining and reconfiguring a triangular cluster near orbits with low eccentricity in real-time. We expect even better results if NTG is coupled with the dynamical systems techniques mentioned above. Moreover, merging this control and dynamics method with image and Earth coverage analysis metrics, will enable mission designers to design optimal observation formations.

Acknowledgment

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