Quadratic Surface Lyapunov Functions in Global Stability Analysis of Saturation Systems

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Abstract

This paper considers quadratic surface Lyapunov functions in the study of global asymptotic stability of saturation systems (SAT), including those with unstable nonlinearity sectors. Quadratic surface Lyapunov functions were first introduced and successfully used to globally analyze stability of limit cycles of relay feedback systems. Later, we showed that equilibrium points of on/off systems could also be globally analyzed using similar ideas. In the state space, on/off systems are composed of a single switching surface. In this paper, we show that global analysis using quadratic surface Lyapunov functions can still be applied to piecewise linear systems (PLS) with more than one switching surface. For that, we consider saturation systems (SAT). A SAT is characterized by an LTI system in feedback with a saturation controller. We present conditions in the form of LMIs that, when satisfied, guarantee *global* asymptotic stability of equilibrium points. A large number of examples was successfully proven globally stable, including systems systems of relative degree larger than one and of high dimension, and systems with unstable nonlinearity sectors, for which classical methods like small gain theorem, Popov criterion, Zames-Falb criterion, IQCs, fail to analyze. In fact, existence of an example of a SAT with a globally stable equilibrium point that cannot be successfully analyzed with this new methodology is still an open problem. The results from this work suggests that other, more complex classes of PLS can be systematically globally analyzed using quadratic surface Lyapunov functions.

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1 Introduction

Piecewise linear systems (PLS) are characterized by a finite number of linear dynamical models together with a set of rules for switching among these models. This captures discontinuity actions in the dynamics from either the controller or system nonlinearities. On one hand, a wide variety of physical systems are naturally modeled this way due to real-time changes in the plant dynamics like collisions, friction, saturations, dead zones, hysteresis, and even more complex systems like walking robots, etc. On the other hand, an engineer can introduce intentional nonlinearities to improve system performance, to effect economy in component selection, or to simplify the dynamic equations of the system by working with sets of simpler equations (e.g., linear) and switch among these simpler models (in order to avoid dealing directly with a set of nonlinear equations). Examples include control of inverted pendulums [1], control of anti-lock brake systems [15], control of missile autopilots [3], control of autopilot of aircrafts [20], auto-tuning of PID regulators using relays [2], etc.

Although widely used and intuitively simple, PLS are computationally hard and very few results are available to analyze most PLS. More precisely, one typically cannot guarantee stability, robustness, and performance properties of PLS designs. Rather, any such properties are inferred from extensive computer simulations. However, in the absence of rigorous analysis tools, PLS designs come with no guarantees. In other words, complete and systematic analysis and design methodologies have yet to emerge.

In the analysis of equilibrium points of PLS, recent research has been concentrating on developing LMI based tools to construct piecewise quadratic Lyapunov functions. These ideas have been proposed and developed by [11], [14], and [10]. Partitioning of the state-space is the key in this approach. For most PLS, construction of piecewise quadratic Lyapunov functions is only possible after a more refined partition of the state space, in addition to the already existent natural state space partition of the PLS. As a consequence, the analysis method is efficient only when the number of partitions required to prove stability is small. As illustrated in an example in [5], however, even for second order systems, the method can become computationally intractable. Also, for high-order systems, it is extremely hard to obtain a refinement of partitions in the state-space to efficiently analyze the PLS. Another disadvantage of finding Lyapunov functions in the state space is that it does not allow to study stability of limit cycles.

The ideas introduced in [7, 8] and used again in [6] were very successful in proving global asymptotic stability of limit cycles and equilibrium points of certain classes of PLS. On the switching surfaces, we efficiently constructed quadratic Lyapunov functions that were used to show that impact maps, i.e., maps from one switching surface to the next switching surface, were contracting in some sense. These results opened the door to the possibility that limit cycles and equilibrium points of more general PLS can be systematically globally analyzed using quadratic surface Lyapunov functions. The main difference between this and previous work, e.g. [11, 14, 10], is that we look for quadratic Lyapunov functions on the switching surfaces instead of quadratic Lyapunov functions on the state space.

The results in [6] represented the first step in analyzing equilibrium points of PLS using quadratic surface Lyapunov functions. In the state space, on/off systems are divided in two partitions by a switching surface. Therefore, the analysis was focused on studying a single switching surface. In the present work, we want to show that quadratic surface Lyapunov functions can also be used to globally analyze PLS with more than two partitions and more than one switching surface.

To demonstrate these ideas, we chose a class of PLS known as saturation systems (SAT). The class of SAT we consider consists of an LTI system in feedback with a saturation. Every time the absolute value of the output of the LTI system exceeds a certain value, a switch occurs and the closed loop system dynamics change. The study of such systems is motivated by the possibility of actuator saturation or constraints on the actuators, reflected sometimes in bounds on available power supply or rate limits. These cannot be naturally dealt within the context of standard (algebraic) linear control theory, but are ubiquitous in control applications. The fact that linear feedback laws when saturated can lead to instability has motived a large amount of research. The well known result which states that a controllable linear system is globally state feedback stabilizable, holds as long as the control does not saturate. In many applications, more often than not, the control is restricted to take values within certain bounds which may be met under closed-loop operation. Because feedback is cut, control saturation induces a nonlinear behavior on the closed-loop system. The problem of stabilizing linear systems with bounded controls has been studied extensively. See, for example, [18, 16, 19] and references therein.

In this paper, we focus on global stability analysis of saturation systems. We are interested in those SAT where the origin is locally stable and is the only equilibrium point. Then, the question is if the origin is also globally asymptotically stable. Rigorous stability analysis for SAT is rarely done. The Zames-Falb criterion [21] can be used when the nonlinearity's slope is restricted, like in this case, but the method is difficult to implement. The Popov criterion can be used as a simplified approach to the analysis, but it is expected to be very conservative for systems of order greater than three. IQC-based analysis [12, 4, 13] gives conditions in the form of LMIs that, when satisfied, guarantee stability of SAT. However, none of these analysis tools can be used when a SAT has an unstable nonlinearity sector.

Here, we propose to construct quadratic Lyapunov functions on the switching surfaces of SAT to show that impact maps, i.e., maps from one switching surface to the next switching surface, associated with the system are contracting in some sense. This, in turn, proves the origin of a SAT is globally asymptotically stable. The search for these quadratic surface Lyapunov functions is done by solving a set of linear matrix inequalities, which can be efficiently done using available computational tools.

This paper is organized as follows. Section 2 starts by formulating the problem. Section 3 presents the main results of this paper followed by some illustrative examples in section 4. Section 5 contains some technical details and, finally, conclusions and future work are discussed in section 6.

2 Problem formulation

The main purpose of this section is to introduce the problem we pretend to solve. We start by defining a saturation system (SAT) followed by some necessary conditions for global stability of a unique locally stable equilibrium point. We then talk about some of the properties of this class of PLS.

Consider a SISO LTI system satisfying the following linear dynamic equations

$$\left\{ \begin{array}{rcl} \dot{x} & = & Ax + Bu \\ y & = & Cx \end{array} \right. \tag{1}$$

where $x \in \mathbb{R}^n$, in feedback with a saturation controller (see figure 1) defined as

$$u(t) = \begin{cases} -d & \text{if } y(t) < -d \\ y(t) & \text{if } |y(t)| \leq d \\ d & \text{if } y(t) > d \end{cases} \tag{2}$$

where $d > 0$ (if $d = 0$ then the system is simply linear). By a solution of (1)-(2) we mean functions (x, y, u) satisfying (1)-(2). Since u is continuous and globally Lipschitz, $Ax + Bu$ is also globally Lipschitz. Thus, the SAT has a unique solution for any initial state.

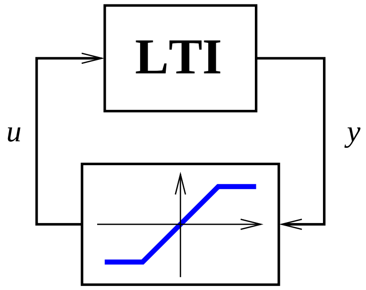

Figure 1: Saturation system

In the state space, the saturation controller introduces two switching surfaces composed of hyperplanes of dimension $n - 1$

$$S = \{x \in \mathbb{R}^n : \; Cx = d\}$$

and

$$\underline{S} = \{x \in \mathbb{R}^n : \; Cx = -d\}$$

On one side of the switching surface S ($Cx > d$), the system is governed by $\dot{x} = Ax + Bd$. In between the two switching surfaces ($|Cx| \leq d$), the system is given by $\dot{x} = Ax + BCx = A_1 x$, where $A_1 = A + BC$. Finally, on the other side of $\underline{S}$ ($Cx < -d$) the system is governed by $\dot{x} = Ax - Bd$. Note that the vector field (1)-(2) is continuous along the switching surfaces since, for any $x \in S$, $A_1 x = (A + BC)x = Ax + Bd$, and for any $x \in \underline{S}$, $A_1 x = Ax - Bd$.

SAT can exhibit extremely complex behaviors. Some SAT may be chaotic, others may have one, three, or a continuum of equilibrium points, or limit cycles, or even some combination of all these behaviors. We are interested in those SAT with a unique locally stable equilibrium point. Only here can a SAT have a globally stable equilibrium point. Several necessary conditions must then be imposed on the system. For instance, it necessary that A has no eigenvalues with positive real part, or otherwise there are initial conditions for which the system will grow unbounded (see for example [17]). A cannot have eigenvalues at zero since that would lead a continuum of equilibrium points. It is also necessary that $A + BC$ is Hurwitz in order to guarantee the origin is locally stable, and $-CA^{-1}B < 1$, so that the origin is the only equilibrium point.

Consider a subset S_+ of S given by

$$S_+ = \{x \in S : \; CA_1 x \geq 0\}$$

This set is important since it tells us which points in S correspond to the first switch of trajectories starting at any x_0 such that $Cx_0 < d$ (see figure 2). In other words, S_+ is the set of points in S that can be reached by trajectories of (1)-(2) when governed by the subsystem $\dot{x} = A_1 x$. In a similar way, define $S_- \subset S$ as

$$S_- = \{x \in S : \; CA_1 x \leq 0\}$$

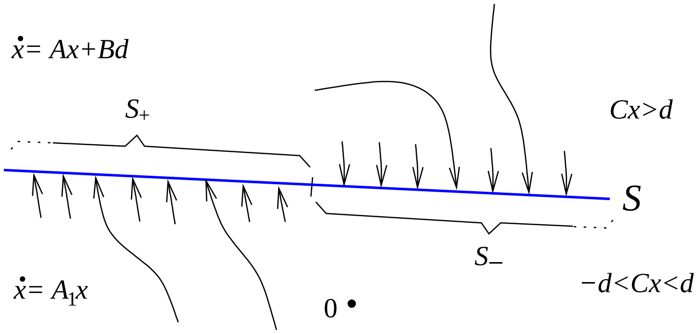

Figure 2: Both sets S_+ and S_- in S

Note that $S = S_+ \bigcup S_-$ and $S_+ \bigcap S_- = \{x \in S : \; CAx = 0\}$. Define also $\underline{S}_+ = -S_+$ and $\underline{S}_- = -S_-$.

Since A_1 must be Hurwitz, there is a set of points in S_- such that any trajectory starting in that set will not switch again and will converge asymptotically to the origin. In other words, let $S^* \subset S_-$ be the set of points x_0 such that the following equations

$$Ce^{A_1 t} x_0 = \pm d$$

do not have a solution for any $t > 0$. Note that this set S^* is not empty. To see this, let $P > 0$ satisfy $PA_1 + A_1' P = -I$. Then, an obvious point in S^* is the point x_1^* obtained from the intersection of S with the level set $x'Px = k$, where $k \geq 0$ is chosen such that the ellipse $x'Px = k$ is tangent to both S and $\underline{S}$ (see figure 3).

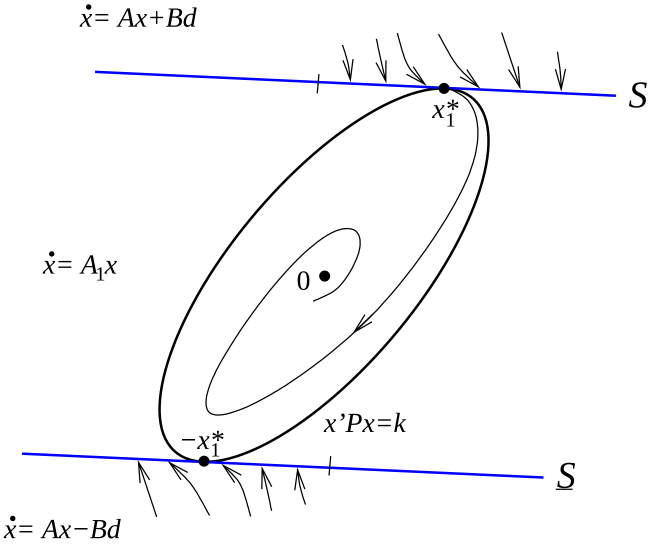

Figure 3: How to obtain x_1^*

The problem we propose to solve is to give sufficient conditions that, when satisfied, prove the origin of a SAT is globally asymptotically stable. The strategy of the proof is a follows. Consider a trajectory starting at some point $x_0 \in S_+$ (see figure 4). Since by assumption $-CA^{-1}B < 1$, the trajectory $x(t)$ will eventually switch at some time $t_1 > 0$, i.e., $Cx(t_1) = d$ and $Cx(t) \geq d$ for $t \in [0, t_1]$. Let $x_1 = x(t_1) \in S_-$. If $x_1 \in S^*$ then the trajectory will not switch again and converges asymptotically to the origin. Since we already know S^* is a stable set, we need to concentrate on those points in $S_- \backslash S^*$ since those are the ones that may lead to potentially unstable trajectories. Here, two scenarios can occur: either the trajectory switches at some point in S or it switches at some point in $\underline{S}$. Let $S_d \subset (S_- \backslash S^*)$ $(S_{-d} \subset (S_- \backslash S^*))$ be the set of points that will eventually switch in S ($\underline{S}$). If $x_1 \in S_d$ $(x_1 \in S_{-d})$ the trajectory switches at some finite time $t_{2a} > t_1$ $(t_{2b} > t_1)$ at $x_{2a} = x(t_{2a}) \in S_+$ $(x_{2b} = x(t_{2b}) \in \underline{S}_+)$. Again, it would switch at $x_{3a} = x(t_{3a})$ $(x_{3b} = x(t_{3b}))$ and so on.

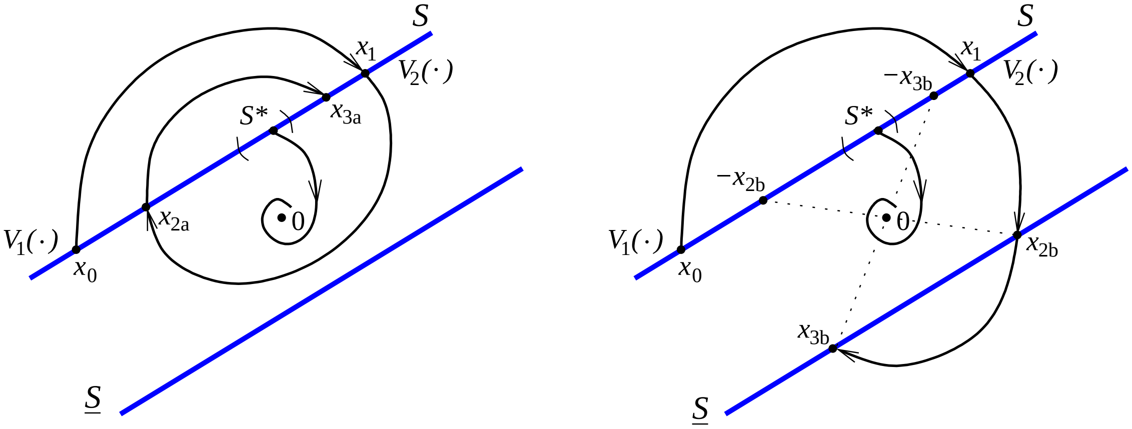

Figure 4: Possible state-space trajectories for a SAT

An interesting property of SAT is their symmetry around the origin. In other words, if $x(t)$ is a trajectory of (1)-(2) with initial condition x_0, then $-x(t)$ is a trajectory of (1)-(2) with initial condition $-x_0$. This means that it is equivalent to analyze the trajectory starting at x_0 or the trajectory starting at $-x_0$. This property is due to the fact that the vector field is symmetric around the origin. If $Cx(t) > d$ then $\dot{x} = Ax + Bd$. Therefore, $-\dot{x} = A(-x) - Bd$ and $C(-x(t)) = -Cx(t) < -d$. If $|Cx(t)| \leq d$ then $\dot{x} = A_1 x$. Hence, $-\dot{x} = A_1(-x)$. Due to this symmetry, whenever a trajectory intersects $\underline{S}$ (like, for example, at x_{2b} in figure 4), for purposes of analysis, it is equivalent to consider the trajectory continuing from the symmetric point around the origin in S ($-x_{2b}$ in figure 4).

As in [6], the idea is to check if x_{3a} or $-x_{3b}$ are closer in some sense to S^* than x_1. If so, this would mean that eventually $x(t_N) \in S^*$, for some N, and prove that the origin is globally asymptotically stable. This is the idea behind the results in the next section.

Before presenting the main results, it is convenient to notice that $x_0, x_1, x_{2a} \in S$ and $x_{2b} \in \underline{S}$ can be parametrized. Let $x_0 = x_0^* + \Delta_0$, $x_1 = x_1^* + \Delta_1$, $x_{2a} = x_0^* + \Delta_{2a}$ and $x_{2b} = -x_0^* + \Delta_{2b}$, where $x_0^*, x_1^* \in S$ and $C\Delta_0 = C\Delta_1 = C\Delta_{2a} = C\Delta_{2b} = 0$. Also, define $x_0^*(t)$ $(x_1^*(t))$ as the trajectory of $\dot{x} = Ax + Bd$ $(\dot{x} = A_1 x)$, starting at x_0^* (x_1^*), for all $t > 0$. Since x_i^* are any points in S, we chose them to be such that $Cx_i^*(t) < d$ for all $t > 0$. The reason for this particular choice of x_0^* and x_1^* is so that $Cx_i^*(t) - d \neq 0$ for all $t > 0$. This will be necessary in proposition 3.1.

This choice of x_0^* and x_1^* is always possible. x_1^* is found as explained above (see figure 3). In this case, x_1^* is given by

$$x_1^* = \frac{P_d^{-1}C'}{CP_d^{-1}C'}d$$

where $P_d > 0$ satisfies $P_d A_1 + A_1' P_d = -I$. In a similar way, x_0^* is given by

$$x_0^* = (d + cA^{-1}Bd)\frac{P_u^{-1}C'}{CP_u^{-1}C'} - A^{-1}Bd$$

where $P_u > 0$ satisfies $P_u A + A' P_u = -I$.

3 Main results

There are three impact maps of interest associated with a SAT. The fist impact map (impact map 1) takes points from S_+ and maps them in S_-. The second impact map (impact map 2a) takes points from $S_d \subset S_-$ and maps them back to S_+. Finally, the third impact map (impact map 2b) takes points from from $S_{-d} \subset S_-$ and maps them in $\underline{S}_+$. Note that the impact maps associated with SAT are, in general, multivalued. Define the sets of expected switching times $\mathcal{T}_1$, $\mathcal{T}_{2a}$, and $\mathcal{T}_{2b}$ as the sets of all possible switching times associated with the respective impact map. In section 5, we show how to get bounds on these sets.

Before presenting the main result, we show that each impact map associated with a SAT can be represented as a linear transformation analytically parametrized by the correspondent switching time.

Proposition 3.1 *Define*

$$w_1(t) = \frac{Ce^{At}}{d - Cx_0^*(t)}\ , \quad w_{2a}(t) = \frac{Ce^{A_1 t}}{d - Cx_1^*(t)}\ , \quad \textit{and} \quad w_{2b}(t) = \frac{Ce^{A_1 t}}{-d - Cx_1^*(t)}$$

Let $H_1(t) = e^{At} + (x_0^*(t) - x_1^*)w_1(t)$*,* $H_{2a}(t) = e^{A_1 t} + (x_1^*(t) - x_0^*)w_{2a}(t)$ *, and* $H_{2b}(t) = e^{A_1 t} + (x_1^*(t) + x_0^*)w_{2b}(t)$*. Then, for any* $\Delta_0 \in S_+ - x_0^*$ *there exists a* $t_1 \in \mathcal{T}_1$ *such that*

$$\Delta_1 = H_1(t_1)\Delta_0$$

Such t_1 *is the switching time associated with* Δ_1*. Similarly, for any* $\Delta_1 \in S_d - x_1^*$ *there exists a* $t_{2a} \in \mathcal{T}_{2a}$ *such that*

$$\Delta_{2a} = H_{2a}(t_{2a})\Delta_1$$

Such t_{2a} *is the switching time associated with* Δ_{2a}*. Finally, for any* $\Delta_1 \in S_{-d} - x_1^*$ *there exists a* $t_{2b} \in \mathcal{T}_{2b}$ *such that*

$$\Delta_{2b} = H_{2b}(t_{2b})\Delta_1$$

Such t_{2b} *is the switching time associated with* Δ_{2b}*.*

We need to show that these three impact maps are contracting in some sense. For that, define two quadratic Lyapunov functions on the switching surface S. Let V_1 and V_2 be given by

$$V_i(x) = x' P_i x - 2x' g_i + \alpha_i \tag{3}$$

where $P_i > 0$, for $i = 1, 2$. Global asymptotically stability of the origin follows if there exist $P_i > 0$, g_i, α_i such that

$$\begin{aligned} V_2(\Delta_1) &< V_1(\Delta_0) && \text{for all } \Delta_0 \in S_+ - x_0^* \\ V_1(\Delta_{2a}) &< V_2(\Delta_1) && \text{for all } \Delta_1 \in S_d - x_1^* \\ V_1(-\Delta_{2b}) &< V_2(\Delta_1) && \text{for all } \Delta_1 \in S_{-d} - x_1^* \end{aligned} \tag{4}$$

Note that in (4) we have mapped the point $\Delta_{2b} \in \underline{S}_+ + x_0^*$ into $S_+ - x_0^*$, taking advantage of the symmetry of the system. Let $P > 0$ on S stand for $x'Px > 0$ for all $x \in S$. As a short hand, in the following result we use $H_{it} = H_i(t)$ and $w_{it} = w_i(t)$.

Theorem 3.1 *Define*

$$\begin{aligned} R_1(t) &= P_1 - H_{1t}'P_2H_{1t} - 2\left(g_1 - H_{1t}'g_2\right)w_{1t} + w_{1t}'\alpha w_{1t} \\ R_{2a}(t) &= P_2 - H_{2at}'P_1H_{2at} - 2\left(g_2 - H_{2at}'g_1\right)w_{2at} - w_{2at}'\alpha w_{2at} \\ R_{2b}(t) &= P_2 - H_{2bt}'P_1H_{2bt} - 2\left(g_2 + H_{2bt}'g_1\right)w_{2bt} - w_{2bt}'\alpha w_{2bt} \end{aligned}$$

where $\alpha = \alpha_1 - \alpha_2$. The origin of the SAT is globally asymptotically stable if there exist $P_1, P_2 > 0$ and g_1, g_2, α such that

$$\begin{cases} R_1(t_1) > 0 & \text{on } S_+ - x_0^* \\ R_{2a}(t_{2a}) > 0 & \text{on } S_d - x_1^* \\ R_{2b}(t_{2b}) > 0 & \text{on } S_{-d} - x_1^* \end{cases}$$

for all expected switching times $t_1 \in \mathcal{T}_1$, $t_{2a} \in \mathcal{T}_{2a}$, and $t_{2b} \in \mathcal{T}_{2b}$.

A relaxation of the constraints on Δ_0 and Δ_1 in the previous theorem results in computationally efficient conditions.

Corollary 3.1 *The origin of the SAT is globally asymptotically stable if there exist $P_1, P_2 > 0$ and g_1, g_2, α such that*

$$\begin{cases} R_1(t_1) > 0 & \text{on } S - x_0^* \\ R_{2a}(t_{2a}) > 0 & \text{on } S - x_1^* \\ R_{2b}(t_{2b}) > 0 & \text{on } S - x_1^* \end{cases} \tag{5}$$

for all expected switching times $t_1 \in \mathcal{T}_1$, $t_{2a} \in \mathcal{T}_{2a}$, and $t_{2b} \in \mathcal{T}_{2b}$.

For each t_1, t_{2a}, t_{2b}, these conditions are LMIs which can be solved for $P_1, P_2 > 0$ and g_1, g_2, α using efficient available software. As we will see in the next section, although these conditions are more conservative than the ones in theorem 3.1, they are already enough to prove global asymptotic stability of many important SAT.

Each condition in (5) depends only on a single scalar parameter. For instance, R_1 depends only on t_1 and not on t_{2a} or t_{2b}. Computationally, this means that when we grid each set of expected switching times, this will only affect one of the conditions in (5). Thus, if we need m_1 samples of $\mathcal{T}_1$, m_{2a} samples of $\mathcal{T}_{2a}$, and m_{2b} samples of $\mathcal{T}_{2b}$, we end up with a total of $m_1 + m_{2a} + m_{2b}$ LMIs. Note that less conservative conditions than those in theorem 3.1 could be obtained. Such conditions, of the form $\bar{R}_1(t_1, t_{2a}) > 0$ and $\bar{R}_2(t_1, t_{2b}) > 0$, would, however, lead to $m_1 \times m_{2a} + m_1 \times m_{2b}$ LMIs, and the analysis problem would easily become computationally intractable. This difference in complexity is even more obvious in the analysis of other, more complex classes of PLS that may require the simultaneous analysis of a large number of impact maps.

The proofs of these results are similar to the ones in [6] and are omitted here.

Conditions (5) are sufficient conditions for the global stability of the origin. These conditions do not take into account that Δ_0, Δ_{1a}, and Δ_{1b} are restricted to S_+, S_d, and S_{-d}, respectively. Using the same ideas from [8, 6], conditions (5) can be improved. For each point $x_0 \in S_+$, there is an associated switching time t_1. Define S_{t_1} as the set of initial conditions $x_0 \in S_+$ such that $y(t) \geq d$ on $[0, t_1]$, and $y(t_1) = d$. This set $S_{t_1} \subset S$ is a convex subset of a linear manifold of dimension $n-2$. Analogously, define $S_{t_{2a}}$ $(S_{t_{2b}})$ as the set of initial conditions $x_{1a} \in S_d$ $(x_{1b} \in S_{-d})$ such that $-d \leq y(t) \leq d$ on $[0, t_{2a}]$, and $y(t_{2a}) = d$ $(y(t) \leq -d$ on $[0, t_{2b}]$, and $y(t_{2b}) = -d)$. Given this, conditions (5) can be improved to

$$\begin{cases} R_1(t_1) > 0 & \text{on } S_{t_1} - x_0^* \\ R_{2a}(t_{2a}) > 0 & \text{on } S_{t_{2a}} - x_1^* \\ R_{2b}(t_{2b}) > 0 & \text{on } S_{t_{2b}} - x_1^* \end{cases} \tag{6}$$

for some $P_1, P_2 > 0$, g_1, g_2, α, and for all expected switching times $t_1 \in \mathcal{T}_1, t_{2a} \in \mathcal{T}_{2a}, t_{2b} \in \mathcal{T}_{2b}$. Approximation to a set of LMIs can be obtained just as in [8, 6].

Note that in many cases, conditions (5) and (6) do not need to be satisfied for all expected switching times. Section 5 shows that bounds on the expected switching times can be obtained. Basically, since $|u| \leq d$ is a bounded input, and when A is Hurwitz, there exists a bounded set such that any trajectory will eventually enter and stay there. This will lead to bounds on the difference between any two consecutive switching times. Let t_{i-} and t_{i+}, $i = 1, 2a, 2b$, be bounds on the minimum and maximum switching times of the associated impact maps. The expected switching times $\mathcal{T}_i$ can, in general, be reduced to a smaller set $[t_{-i}, t_{i+}]$. Conditions (5) and (6) can then be relaxed to be satisfied only on $[t_{i-}, t_{i+}]$ instead on all $t_i \in \mathcal{T}_i$. See section 5 for details.

4 Examples

The following examples were processed in `matlab` code. The latest version of this software is either available at [9] or upon request. Before we present the examples, we briefly explain the `matlab` function that we developed. The input to this function is a transfer function of an LTI system together with a parameter $d > 0$. If the SAT is proven globally stable, the `matlab` function returns the parameters of the two quadratic surface Lyapunov functions (3). We then confirm conditions (5) are satisfied by plotting the minimum eigenvalues of each $R_i(t)$ on $[t_{i-}, t_{i+}]$, and showing that these are indeed positive in those intervals.

Before moving into the examples, it is important to explain how the bounds $[t_{i-}, t_{i+}]$ on the expected switching times are found. First, notice that $t_{1-} = t_{2a-} = 0$. Zero switching time for the first impact map $\Delta_0 \rightarrow \Delta_1$ and the second impact map $\Delta_1 \rightarrow \Delta_{2a}$ correspond to points in S such that $CA_1x = 0$. At those points, the Lyapunov functions (3) must be continuous since this is the only way

$$\begin{cases} V_2(\Delta_1) \leq V_1(\Delta_0) \\ V_1(\Delta_{2a}) \leq V_2(\Delta_1) \end{cases}$$

can be satisfied simultaneously, for all $\Delta_0, \Delta_1, \Delta_{2a} = \Delta_0$ such that $x_0^* + \Delta_0 = x_1^* + \Delta_1 = x$ and $CAx = 0$. Therefore, for those points we need $V_1(\Delta_0) = V_2(\Delta_1)$. This imposes certain restrictions on $P_1, P_2 > 0$, g_1, g_2, and α. The analysis of zero switching time for these two impact maps is similar to the case of on/off systems [6, 5]. See [5, section 6.7.2] for details.

As for the map $\Delta_1 \rightarrow \Delta_{2b}$, zero switching never occurs since there is a "gap" between S and $\underline{S}$, resulting in a nonzero switching time for every trajectory starting in S_{-d}. For certain large values of $\|\Delta_1\|$, however, the switching times can be made arbitrarily small. But, when A is Hurwitz, we know all system trajectories eventually enter an invariant bounded set, just like in relay feedback systems [8]. This can be seen from the fact that the open loop system is stable and $|u| \leq d$ is bounded. In this invariant bounded set, switching times for the impact map $\Delta_1 \rightarrow \Delta_{2b}$ cannot be made arbitrarily small, and a lower bound can be found. Using the same ideas, upper bounds on expected switching times for all impact maps can be found. All the details are in section 5. The case when A has imaginary eigenvalues is currently under investigation.

Example 4.1 Consider the SAT on the left of figure 5 with $d = 1$. It is easy to see the origin of this system is locally stable. The question is if the the origin is also globally asymptotically stable.

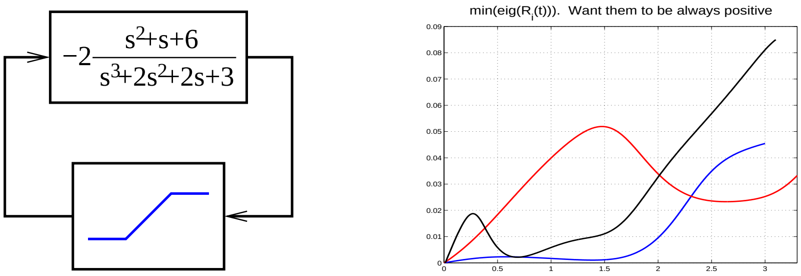

Figure 5: 3^{rd}-order system with unstable nonlinearity sector

Using conditions (5), we show that the origin is in fact asymptotically globally stable. The right side of figure 5 illustrates this fact: the minimum eigenvalue of each condition (5) is positive on its respective set of expected switching times. The expected switching times in this example are approximately $\mathcal{T}_1 = (0, 3)$, $\mathcal{T}_{2a} = (0, 3.3)$, and $\mathcal{T}_{2b} = (0, 3.1)$. For instance, if $t_1 \geq 3$, there is no point in S_+ with switching time equal to t_1.

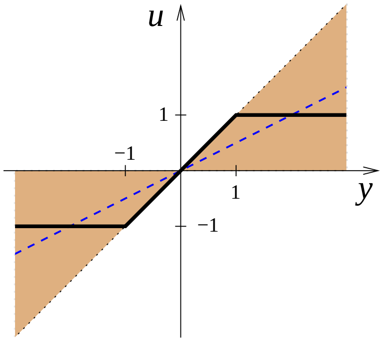

Figure 6: Saturation controller versus constant gain of 1/2 (dashed)

Note that this system has an unstable nonlinearity sector. If the saturation is replaced by a linear constant gain of 1/2, the system becomes unstable (see figure 6). This is very interesting since it tells us that classical analysis tolls like small gain theorem, Popov criterion, Zames-Falb criterion, and integral quadratic constraints, fail to analyze SAT of this nature. ■

Example 4.2 Consider the SAT in figure 7 with $d = 1$ and $k > 0$. The origin of the SAT is locally stable for any $k > 0$.

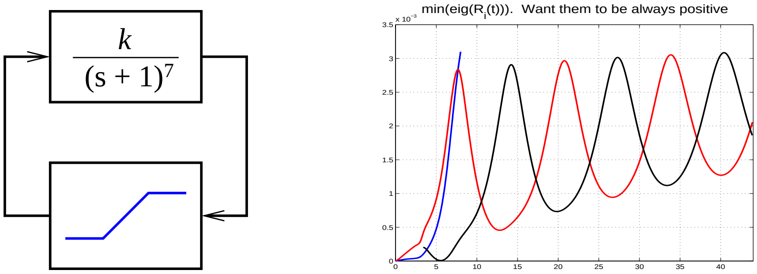

Figure 7: System with relative degree 7 (left); global stability analysis when $k = 2$ (right)

Note that $\|Ce^{At}B\|_{\mathcal{L}_1} = k$, which means the small gain theorem can only be applied when $k < 1$.

Let $k = 2$. The right side of figure 7 shows how conditions (5) are satisfied in some intervals (t_{i-}, t_{i+}), $i = 1, 2a, 2b$. The intervals (t_{i-}, t_{i+}) are bounds on the expected switching times. Such bounds are such that if conditions (5) are satisfied on (t_{i-}, t_{i+}), then the system is globally asymptotically stable. For details on how to find these bounds see section 5. ■

5 Technical Details: Bounds on Switching Times

In this section, we will talk about computational aspects related to finding $P_i > 0$, g_i, and α in (5) or (6). For many SAT, the sets of expected switching times are equal to the set $[0, \infty)$. Thus, in general, it is impossible to check directly if the stability conditions (5) or (6) are satisfied for all expected switching times. An alternative is to find some intervals (t_{i-}, t_{i+}) such that if (5) or (6) are satisfied in those intervals, then stability follows.

In [8], we showed that in the case of RFS, there is a bounded invariant set where every trajectory will eventually enter. Hence, bounds on the expected switching times could be found by computing bounds on switching times of trajectories inside that bounded invariant set. This same idea can be used here whenever A is Hurwitz. In fact, since $u = \pm d$ is a bounded input, a bounded and invariant set such that any trajectory will eventually enter can be found. This will lead to bounds on the difference between any two consecutive switching times. This way, the search for $P_i > 0$, g_i, and α in (5) and (6) becomes restricted to $0 \leq t_{i-} < t < t_{i+} < \infty$, $i = 1, 2a, 2b$.

As explained before, $t_{1-} = t_{2a-} = 0$ since the associated impact maps are defined on the same switching surface, and are allowed to have zero switching time. We then focus on upper bounds for all impact maps and on the lower bound t_{2b-} of impact map $2b$. Notice there are many ways to finds such bounds and the method we propose next is not unique, and can surely be improved.

Before we find such bounds, we need to show there is a particular bounded invariant set such that any trajectory will eventually enter and stay there. The following proposition is similar to proposition 7.1 in [8]. Thus, the proof is omitted here. In this result, $\|f(t)\|_{\mathcal{L}_1}$

stands for

$$\|f(t)\|_{\mathcal{L}_1} = \int_0^\infty |f(t)|\, dt$$

Proposition 5.1 *Consider the system $\dot{x} = Ax + Bu$, $y = Fx$, where A is Hurwitz, $u(t) = \pm d$, and F is a row vector. Then*

$$\limsup_{t\to\infty} |Fx(t)| \leq d\|Fe^{At}B\|_{\mathcal{L}_1}$$

As a remark, if $F = C$ and $\|Fe^{At}B\|_{\mathcal{L}_1} < 1$, it follows the origin is globally asymptotically stable. When $\limsup_{t\to\infty} |Cx(t)| < d$, eventually all trajectories enter the set $\{x| \; |Cx| < d\}$, where the system is linear and stable. Note that this remark also follows from the well known small-gain theorem.

We first focus our attention on upper bounds of the switching times t_{i+}, starting with t_{1+}. A trajectory $x(t)$ starting at $x_0 \in S_+$ is given by $x(t) = e^{At}(x_0 + A^{-1}Bd) - A^{-1}Bd$. Thus, the output $y(t) = Cx(t)$ is given by

$$y(t) = Ce^{At}(x_0 + A^{-1}Bd) - CA^{-1}Bd$$

Since we are assuming $-CA^{-1}Bd < d$, and A Hurwitz, it is easy to see that $y(t)$ cannot remain larger than d for all $t > 0$. For any initial condition $x_0 \in S_+$, $Ce^{At}(x_0 + A^{-1}Bd) \to 0$ as $t \to \infty$, which means $y(t) = d$ for some t. Thus, a switch must occur in finite time. Since for a sufficiently large enough time t, $x(t)$ enters a bounded invariant set (from the above proposition), an upper bound on this switching time t_{1+} can be obtained.

Proposition 5.2 *Let $t_{1+} > 0$ be the smallest solution of*

$$\int_{t_{1+}}^\infty \left| Ce^{At}B \right| dt + |Ce^{At_{1+}}A^{-1}B| \leq (CA^{-1}B + 1)$$

If t_a and t_b are sufficiently large consecutive switching times of the first impact map then $|t_a - t_b| \leq t_{1+}$.

Next, we find upper bounds on the expected switching times of impact maps $2a$ and $2b$. The idea here is to find the minimum $t_2 \geq 0$ such that

$$|y(t)| = \left| Ce^{A_1 t}x_0 \right| \leq d$$

for all $t \geq t_2$ and all x_0 in the bounded invariant set. In this derivation, $t_{2a+} = t_{2b+} = t_2$.

Proposition 5.3 *Let $t_2 > 0$ be the smallest solution of*

$$\int_0^\infty \left| Ce^{A_1 t_2}e^{At}B \right| dt \leq 1 \tag{7}$$

If t_a and t_b are sufficiently large consecutive switching times of impact maps $2a$ or $2b$, then $|t_a - t_b| \leq t_2$, and $t_{2a+} = t_{2b+} = t_2$.

We now focus on the lower bound on the expected switching times of impact map $2b$, i.e, t_{2b-}. Remember that if $x_0 \in S_+$, then $y(0) = d$. Since $d > 0$, it must be true that $y(t) > -d$ at least in some interval $(0, \epsilon)$. Basically, the time it takes to go from S to $\underline{S}$ must be always nonzero. The next result shows that when a trajectory enters the bounded invariant set characterized above, ϵ cannot be made arbitrarily small. Thus, a lower bound on the time it takes between two consecutive switches from S to $\underline{S}$ can be obtained.

Proposition 5.4 *Let* $k_{dd} = \|CA_1^2 e^{At} B\|_{\mathcal{L}_1}$*, and* $k_{dl} = \|CA_1 e^{At} B\|_{\mathcal{L}_1}$ *and define*

$$t_{21} = \frac{2}{\sqrt{k_{dd}}}, \quad t_{22} = \frac{2}{k_{dl}}$$

Let $t_{2b-} = \max\{t_{21}, t_{22}\}$*. If* t_a *and* t_b *are sufficiently large consecutive switching times of impact map 2b, then* $|t_a - t_b| \geq t_{2b-}$*.*

6 Conclusion

This paper confirms the idea that global stability analysis of equilibrium points and limit cycles of certain classes of piecewise linear systems can be done using impact maps and quadratic surface Lyapunov functions. In particular, this paper showed the success of this methodology in global stability analysis of PLS with more than one switching surface.

In [7, 8] and [6] we demonstrated how this approach is powerful in globally analyzing limit cycles of relay feedback systems and equilibrium points of on/off systems, respectively. Here, we demonstrated that similar ideas can be used to check if equilibrium points of saturation systems are globally asymptotically stable. Impact maps can be proven quadratically stable by constructing quadratic Lyapunov functions on switching surfaces. The search for quadratic surface Lyapunov functions is efficiently done by solving a set of LMIs.

With this new results, a large number of examples was successfully proven globally stable. These include systems of relative degree larger than one and of high dimension, and systems with unstable nonlinearity sectors, for which all classical fail to analyze. In fact, existence of an example with a globally stable equilibrium point that could not be successfully analyzed with this new methodology is still an open problem.

There are still many open problems following this work. The main question is how to use quadratic surface Lyapunov functions to systematically globally analyze larger and more complex classes of PLS. Other topics of ongoing research include the development of analysis tools, based on quadratic surface Lyapunov functions, to analyze robustness and performance of certain classes of PLS.

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