INFORMATION FLOW AND COOPERATIVE CONTROL OF VEHICLE FORMATIONS

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Abstract: Vehicles in formation often lack global information regarding the state of all the vehicles, a deficiency which can lead to instability and poor performance. In this paper, we demonstrate how exchange of minimal amounts of information between vehicles can be designed to realize a dynamical system which supplies each vehicle with a shared reference trajectory. When the information flow law is placed in the control loop, a separation principle is proven which guarantees stability of the formation and convergence of the information flow law regardless of the information flow topology.

Keywords: decentralized control, graph theory, vehicles

1. INTRODUCTION

Recent years have seen the emergence of control of vehicle formations as a topic of significant interest to the controls community. Applications span a wide range, including mobile robotics, traffic control, satellite clusters and UAV formations. A recent study (Air Force Scientific Advisory Board, 1995) identified this area as needing fundamentally new control paradigms.

Central to a discussion of cooperative vehicle control is a determination of the nature of the *information flow* throughout the formation. We will distinguish between two types of information flow: *sensed information*, meaning the ability of a single vehicle to sense some information (e.g. relative position) about another vehicle in a way which involves no action on the part of that vehicle, and *transmitted information*, meaning transfer of information between two vehicles which requires some action on the part of both the transmitter and recipient of that information. Sensing and transmission are the means by which each vehicle acquires the information necessary to perform its task within the formation. The sensing and transmitted information topologies are themselves dynamic, meaning they are subject to change due to changes in the formation itself or due to outside influences.

In Fax and Murray (2002), we considered the effect of the sensing network topology on stability of a vehicle formation. The sensing paths were modeled as a graph, and stability concerns arose when cycles were present in the graph. We proved a Nyquist-like criterion which used the eigenvalues of the Laplacian matrix of the graph in determining stability of the formation. In that paper, no information exchange between vehicles was considered. In this paper, we extend our other work by focusing on improving performance through the exchange of information between vehicles.

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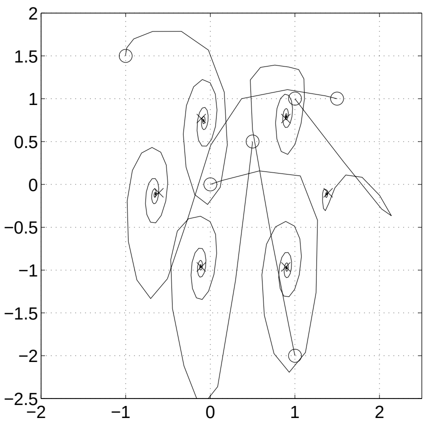

Fig. 1. Vehicle Motion Without Information Flow.(see text for explanation)

From a performance perspective, sparseness in the sensing graph means that each vehicle has only a limited picture of the behavior of the formation, and cannot therefore make informed control decisions. We will demonstrate an information flow law which realizes a discrete-time dynamic system whose convergence enables the vehicles to achieve *consensus* regarding the formation center. Our method is driven by the need to ensure stability of the formation and robustness to changes in the information flow topology. As such, our methodology differs from traditional approaches to achieving consensus (Lynch, 1996).

2. INFORMATION FLOW IN VEHICLE FORMATIONS

2.1 *Motivating Example*

To motivate our discussion of information flow, we consider the following sample problem. Six vehicles, whose dynamics are double integrators in the plane, are asked to take up positions on the points of a regular hexagon relative to one another, and each vehicle can measure its position relative to some (small) subset of the other vehicles. Using the techniques in Fax and Murray (2002), stability of the formation for a given controller and graph can be verified. Figure 1 shows the trajectories the vehicles follow, beginning at the 'o' locations and ending in the 'x' locations. The vehicles follow circuitous trajectories to their final destination, since no vehicle has a clear picture of what the overall formation is doing. In this case, the piece of information each vehicle needs is some sense of the formation center. Our goal will be to devise a means of information exchange which enables the vehicles to arrive at a consensus as to the formation center, and which is robust to uncertainty and changes in the various network topologies.

2.2 *An Information Flow Paradigm*

In Fax and Murray (2002), we assumed that sensed information was available instantaneously, and we used a continuous-time model of the vehicle dynamics. In this paper, we will assume that information takes a fixed time T to travel between vehicles. To facilitate analysis, we also model our vehicle as a discrete time dynamical system:

$$\begin{aligned} x^i_{k+1} &= P_A x^i_k + P_B u^i_k \\ y^i_k &= P_C x^i_k + P_D u^i_k \end{aligned} \tag{1}$$

where k is the time step of duration T and i is the vehicle index. As in Fax and Murray (2002), the error signal used by each vehicle is

$$z^i_k = \frac{1}{|J^S_i|} \sum_{j \in J^S_i} y^i_k - y^j_k, \tag{2}$$

meaning an average of the relative error measurement available to each vehicle. The (presumed non-empty) index set J^S_i represents the set of vehicles visible to vehicle i, and we can form a directed graph based on those sets. [2] When Equation (2) is represented as a matrix, it takes the form

$$z_k = L_{(n)} y_k, \tag{3}$$

where L is the Laplacian of the graph, and the (n) subscript indicates that each element of L is replaced with I_n for dimensional compatibility. The Laplacian is defined as $I - D^{-1}A$, where A is the adjacency matrix of the graph, and D has the in-degree of each vertex along the diagonal (Chung, 1997). In the future, the subscripts will be omitted, and dimensional compatibility will be assumed. Note that the stability results of Fax and Murray (2002) can be reproduced for discrete time systems if one uses the discrete Nyquist criterion rather than the continuous one.

Any information flow consists of vehicles receiving a transmission from other vehicles and performing some computation using that information, information from previous transmissions, and sensed information. Each vehicle then transmits the results of their computation to other vehicles. We can view this process as a discrete-time dynamical system where the states are the information at each vehicle. The ability of a vehicle to receive transmissions from another vehicle can be captured in index sets J^T_i, which defines a second directed graph. In this paper, we will assume that $J^T_i = J^S_i$, and omit the superscript.

[2] In Fax and Murray (2002) we omitted the superscript S, which we include here to identify it as the sensed information index set, as opposed to the transmitted information index set.

The information flow law we are going to investigate will take the following form:

$$p^i_{k+1} = z^i_k + \frac{1}{|J_i|}\sum_{j\in J_i} p^j_k \qquad (4)$$

or, in vector form:

$$p_{k+1} = Gp_k + Ly_k, \qquad (5)$$

where $G = D^{-1}A = I - L$. Each vehicle's new information is thus based on the average of the sum of sensed and transmitted information from other vehicles.

2.3 *Properties of the Information Flow Law*

We now analyze stability and convergence properties of the information flow law, making use of ideas from Perron-Frobenius theory. As discussed in detail in Fax and Murray (2002), G is a non-negative matrix whose Perron root is 1 and whose eigenvalues must lie in the unit circle. The graph is termed *strongly connected* if any two nodes can be joined by a path, and *aperiodic* if the lengths of cycles in the graph do not have a greatest common divisor other than 1. We assume the graph to be strongly connected[3], which implies that G has positive left and right Perron eigenvectors e_l, e_r. Define $E = e_r e_l^T$, where e_l, e_r are chosen such that $e_r^T e_l = 1$, and let $\bar{G} = G - E$. We will make use of the following two results (Horn and Johnson, 1985):

Lemma 1. $G^n = E + \bar{G}^n$.

Lemma 2. The eigenvalues of $\bar{G}$ are the eigenvalues of G with the Perron eigenvalue replaced with a zero eigenvalue.

We now derive the following result:

Theorem 3. Suppose the directed graph G is strongly connected and aperiodic, and let the input y_k be fixed in time. The steady state value of the dynamical system in Equation (5), when $p_0 = 0$, is

$$p^i_{ss} = y^i - \sum_{j=1}^{N} e^i_l y^i \qquad (6)$$

where e^i_l is scaled so that $\sum e^i_l = 1$.

PROOF. We consider the evolution of Equation (5):

$$p_k = G^k p_0 + \left(\sum_{j=0}^{k} G^j\right) Ly \qquad (7)$$

We assume that $p_0 = 0$, and we wish to find $p_{ss} = \lim_{k\to\infty} p_k$, if such a limit exists. Substituting into Equation (7) via Lemma 1, we can replace G^j with $E+\bar{G}^j$. Recalling that $E = e_r e_l^T$,and that L shares eigenvectors with G, we see that e_r and e_l are the eigenvectors of L corresponding to the zero eigenvector. Therefore, $EL = e_r e_l^T L = e_r 0 = 0$, and we can rewrite p_k as

$$p_k = \left(\sum_{j=0}^{k} \bar{G}^j\right) Ly \qquad (8)$$

Because G is assumed strongly connected and aperiodic, all non-Perron eigenvalues of G have modulus strictly less than one (see Fax and Murray (2002)). Therefore, by Lemma 2, we see that $\rho(\bar{G}) < 1$. Equation (8) therefore converges and can be written:

$$p_{ss} = (I - \bar{G})^{-1} Ly \qquad (9)$$
$$= (L + E)^{-1} Ly \qquad (10)$$
$$= (I - (L + E)^{-1} E)y \qquad (11)$$

Now $Le_r = 0$, and $Ee_r = (e_r e_l^T)e_r = e_r(e_l^T e_r) = e_r$, so $(L + E)e_r = e_r \Rightarrow (L + E)^{-1} e_r = e_r$, and the above equation can be rewritten

$$p_{ss} = (I - e_r e_l^T)y \qquad (12)$$

Because the rows of G sum to one, $e_r = 1^T$, and e_l is scaled such that $\sum e^i_l = 1$. The columns of E are therefore constant, and the rows are each e_l^T. Therefore, Equation (12) is equivalent to Equation (6). Proven.

We see that the information flow supplies each vehicle with a formation center defined by a graph-dependent weighting. We now consider a more general form for the information flow law:

$$\begin{aligned} q_{k+1} &= \sum_{i=0}^{R} a_i q_{k-i} + \sum_{i=0}^{R} b_i G q_{k-i} + Ly_k \\ p_k &= \sum_{i=0}^{R} c_i q_{k-i} \end{aligned} \qquad (13)$$

In this version, we are computing our current information using information from several previous time steps. (This formulation can also be used to account for the presence of additional transmission delays.) As in the previous case, we wish to determine both stability and the convergence properties.

[3] This assumption can be relaxed if a path exists from any node to a single strongly connected subgraph.

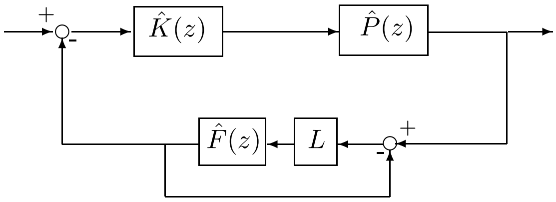

Fig. 2. Block diagram of Information Flow

Theorem 4. The system in Equation (13) is (neutrally) stable if the transfer function

$$F(z) = \frac{\sum_{i=0}^{R} b_i z^{R-i}}{z^{R+1} - \sum_{i=0}^{R} (a_i + b_i) z^{R-i}} \tag{14}$$

is (neutrally) stable and its Nyquist plot avoids encirclement of the negative inverse of any of the nonzero eigenvalues of L.

PROOF. We can take the z-transform of Equation (13), setting aside y_k, and rewrite it as follows:

$$zq(z) = \sum_{i=0}^{R} (a_i + b_i G)\, z^{-i} q(z) \tag{15}$$

$$= \sum_{i=0}^{R} (a_i + b_i - b_i L) z^{-i} q(z) \tag{16}$$

or, if we collect terms not including L and multiply both sides by z^R,

$$q(z) = \frac{-\sum_{i=0}^{R} b_i z^{R-i}}{z^{R+1} - \sum_{i=0}^{R} (a_i + b_i) z^{R-i}} Lq(z). \tag{17}$$

This equation is seen to be a negative feedback loop with $F(z)$ and L in the forward path, equivalent to the lower loop in Figure 2. This is the same format as the system of vehicle formations examined in Fax and Murray (2002), where it was shown that the stability of this system is given by the Nyquist criterion stated above. Because one set of eigenvalues of this system corresponds to the open-loop dynamics, this system can be at best neutrally stable if $F(z)$ is itself neutrally stable. Proven.

We now wish to determine the steady-state performance of a given information flow law. We will set $c_i = b_i$, which will ensure that the information flow loop has unity DC gain and will be useful in proving stability in Theorem 6. We further assume that $F(z)$ has all poles on the interior of the unit circle with the possible exception of a pole at 1, and that the polynomial $\sum_{i=0}^{R} a_i z^{R-i}$ also has roots in the interior of the unit circle.

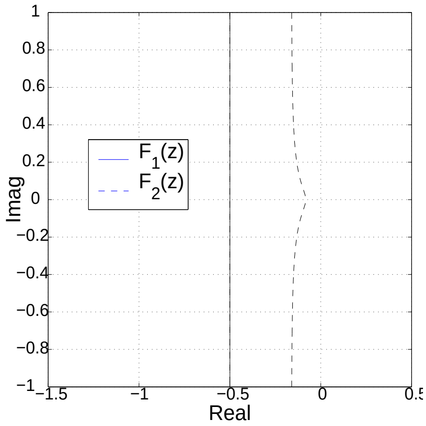

Fig. 3. Information Flow Law Nyquist Plots.

Theorem 5. If $F(z)$ stabilizes L in the sense of Theorem 4, and under the above assumptions,

$$p_{ss} = c\left(I - cE - (1-c)\left(I - c\bar{G}\right)^{-1} G\right) y \tag{18}$$

where $a = \sum_{i=0}^{R} a_i$, $b = \sum_{j=0}^{R} b_j$, and $c = \frac{b}{1-a}$.

The proof, which can be found in Fax and Murray (2001), uses the Final Value Theorem along with manipulations similar to those in Theorem 3.

Note that $c = 1$ corresponds to $a + b = 1$, which implies that the system has a pole at 1. When $c = 1$, we recover the same steady-state result we had derived earlier. When $c < 1$, the steady-state is offset by an additional term. Note that when $c = 1$, the vehicles all agree on the location of the formation center (expressed in each vehicle's coordinates), while when $c < 1$, they do not. We can say that when $c = 1$, the vehicles achieve *consensus* on formation center, whereas when $c \neq 1$ they do not. The coincidence of the kernel of L with the Perron eigenvector of G prevents secular growth in p. However, the presence of noise or sensor errors has the potential to introduce drift.

To understand the effects of shaping the information flow, we will consider two examples. The first information flow law is in Equation (5). In this case, following Equation (14) $F_1(z) = \frac{1}{z-1}$. The second is given by $a_0 = 1.0625$, $a_1 = -0.2313, b_0 = 0.1875$, $b_1 = -0.0188$, which corresponds to $F_2(z) = \frac{0.1875(z-0.1)}{(z-0.25)(z-1)}$. The pole at 1 means that $c = 1$ in both cases. Figure 3 shows the Nyquist plot for these two cases. The solid line, representing $F_1(z)$, lies along the -0.5 vertical. Points on that line correspond to periodic graphs (see Fax and Murray (2002)), which confirms Theorem 3. The dashed line, representing $F_2(z)$, lies entirely to the right of the -0.5 vertical, meaning that it will stabilize any graph. For sparse, though aperiodic graphs, $F_2(z)$ displays a much smoother transient response.

$$
T^{-1}\Psi T = \begin{pmatrix} P_A & 0 & 0 & 0 & 0 \\ -CLP_C & M_a + CGM_b & 0 & 0 & 0 \\ 0 & -K_B\Delta M_b & K_A + K_B P_D \Delta K_C & K_B \Delta P_C & 0 \\ 0 & -P_B K_D \Delta M_b & P_B \Delta K_c & P_A + P_B K_D \Delta K_C & 0 \\ CLP_C & -\phi M_b & CP_D \Delta K_C & C\Delta P_C & M_a \end{pmatrix} \tag{27}
$$

3. INFORMATION FLOW IN THE LOOP

The information flow law supplies each vehicle with the information it cannot sense: a (graph-dependent) formation center about which to do control. A natural strategy is to use p as the input to the controller $K(z)$ rather than y, as shown in Figure 2. As before, we can analyze stability with respect to uncertainties in the graph by isolating L and applying the Nyquist criterion. In this case, one determines stability by analyzing the Nyquist plot of

$$
F(z)(1 + P(z)K(z)). \tag{19}
$$

For a given plant and controller, the information flow loop can be designed to provide stability. However, care must be taken in interpreting the stability margins derived from this plot. The gain and phase margins of this plot do not correspond to uncertainties in the plant in the typical fashion due to the location of $P(z)$ in the transfer function, but to uncertainties in L. Small variations in $P(z)$ can produce unexpected perturbations of the Nyquist plot. In other words, the coupling between the dynamics of the information flow, controller, and plant can produce unexpected results.

The information flow law presented earlier is necessarily reactive; it does not anticipate the motion of the cluster. A logical means of improving performance of the information flow loop is to supply the information flow loop with information with feedforward information. In this section, we shall explore augmentations to the information flow loop which follow this paradigm.

Recalling that the information represents an averaged position of the vehicles' positions, a logical choice for a feedforward signal is the anticipated change in vehicle position. This can be calculated by using each vehicles' control signal $u(z)$ as the input to a model of the plant $\tilde{P}(z)$, and differencing the output. The resulting signal is then transmitted in addition to the signal q and used by each vehicle as a correction term to p. Of course, this equation is only current if the control signal is delayed by a time step before application to the plant to allow a time step for the information to reach the other vehicles. Alternatively, each vehicle could delay the use of its sensed information until it receives the transmitted information from that vehicle. Generally, the feedforward correction term will take on the form

$$
w(z) = H(z)\tilde{P}(z)u(z). \tag{20}
$$

When $H(z)$ is chosen properly, the following result can be derived:

Theorem 6. Choose $H(z) = (F(z) + 1)^{-1}$ and suppose the feedback interconnection of $P(z)$ and $K(z)$ is well-posed. Then the formation is stabilized if $F(z)$ stabilizes L and $K(z)$ stabilizes $P(z)$.

Using our definition of $F(z)$, we can write $H(z)$ as

$$
H(z) = \frac{z^{R+1} - \sum_{i=0}^{R}(a_i + b_i)z^{R-i}}{z^{R+1} - \sum_{i=0}^{R} a_i z^{R-i}}. \tag{21}
$$

The assumption that the coefficients of the numerator sum to 1 imply that $H(z)$ has a zero at 1, which corresponds to differencing the input signal. Note that $H(z)$ is stable by the assumptions on Theorem 5.

PROOF. We prove the presence of a separation principle for the system of equations, through the use of a change of coordinates. The state-space equations of motion for the plant are given in Equation (1). The predictor $\tilde{P}(z)$ is presumed to be identical to the plant $P(z)$, and has the same equations of motion with x, y replaced by $\tilde{x}, \tilde{y}$. The dynamics of the controller will be represented as

$$
\begin{aligned} v_{k+1} &= K_A v_k + K_B p_k \\ u_k &= K_C v_k + K_D p_k. \end{aligned} \tag{22}
$$

Defining

$$
M_a = \begin{pmatrix} 0 & I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I \\ a_R I & a_{R-1} I & \dots & a_0 I \end{pmatrix}, \tag{23}
$$

$M_b = \begin{bmatrix} b_R I & \dots & b_0 I \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 0 & \dots & I \end{bmatrix}^T$, we can write the information flow law of Equation (13) with the feedforward term added in vector form as

$$
\begin{aligned} \bar{q}_{k+1} &= (M_a + CGM_b)\bar{q}_k + CLy_k + CGw_k \\ p_k &= M_b \bar{q}_k + w_k \end{aligned} \tag{24}
$$

and the feedforward correction term of Equation (20) as

$$
\begin{aligned} \bar{r}_{k+1} &= M_a \bar{r}_k + C\tilde{y}_k \\ w_k &= -M_b \bar{r}_k + \tilde{y}_k. \end{aligned} \tag{25}
$$

If one solves Equations (1),(22),(24),(25) for the states, the resulting system can be written as

$$X_{k+1} = \Psi X_k. \tag{26}$$

If the states are chosen according to the linear combination $X_k = [x_k - \tilde{x}_k, \bar{r}_k - \bar{p}_k, v_k, \tilde{x}_k, \bar{p}_k]^T$, then Ψ takes the form found in Equation (27), where $\phi = C(P_D K_D \Delta + G)$, and $\Delta = (I - P_D K_D)^{-1}$, which is invertible by assumption of well-posedness of the interconnection.

Stability is thus determined by stability of the blocks along the diagonal of Ψ. The first, P_A, is neutrally stable by assumption. It is also assumed that M_a, M_b are chosen to (at least neutrally) stabilize the second block, $M_a + CGM_b$. The third block along the diagonal, which comprises the third and fourth columns/rows, is stable if $K(z)$ stabilizes $P(z)$. The final block is stable by the assumptions of Theorem 5. Proven.

If $c \neq 1$, the vehicles' final positions will incorporate the errors of Equation (18) as well. The position of the vehicles will also depend on the ability of the information flow law to track the natural motion of the vehicles. When the vehicles' natural motion displays secular drift or oscillation, the quality of the reference signal will depend on the ability of the information flow law to track signals at the relevant frequencies. Additionally, note that the motion of the formation is sensitive to mismatches between initial conditions of the vehicle and predictor. This can lead to drift of the cluster due to mismatches in velocity. It may be possible to improve upon this through the use of an observer which will prevent the vehicle and predictor from diverging. The zero at 1 in $H(z)$ corresponds to differencing the input, which generally amplifies signal noise. However, the input to $H(z)$ is derived by integrating u, so no net differencing when computing w.

We now return to the sample problem posed earlier. Figure 4 shows the trajectories when the information flow loop is enabled. Two trajectories are overlaid: the solid line shows the trajectories when the information flow loop and the vehicle controller are enabled simultaneously, and the dashed line shows the trajectories when the information loop runs for 1 second prior to enabling the controller. In the first case, the trajectories are much smoother than when no information is used, but still show some curving caused by application of control prior to convergence of the information loop. In the latter, the vehicles follows straight lines to their destinations. The final destinations in each case do not differ, despite lack of absolute position information, due to the decoupling of the information flow loop from the predicted motion of the vehicles.

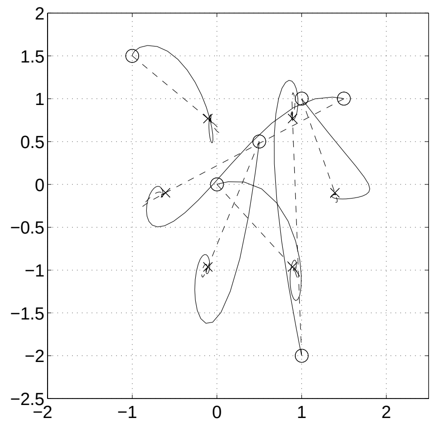

Fig. 4. Vehicle Motion With Information Flow (See text for explanation).

4. CONCLUSIONS

Our results in this paper rely on two key ideas. The first is the use of dynamical systems as a paradigm for understanding information exchange between vehicles, and the design of a dynamical system which enables the vehicles to achieve consensus on the formation center. A key feature of this approach is that no vehicle need have knowledge of the global structure of the formation in order to play its part. This renders the approach highly robust to changes in that structure. The second is the use of feedforward compensation to render the sensed and transmitted information timely. Naturally, this depends on a good model of the vehicle dynamics. Future research will focus on extending these core notions to a broader range of plants, controllers, and objectives, as well as issues such as variable time delays in transmission, as might occur in control over networks.

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