GRAPH LAPLACIANS AND STABILIZATION OF VEHICLE FORMATIONS

J. Alexander Fax *,[1] **Richard M. Murray** *

* *Engineering and Applied Science*
California Institute of Technology
Pasadena, CA 91125

Abstract: Control of vehicle formations has emerged as a topic of significant interest to the controls community. In this paper, we merge tools from graph theory and control theory to derive stability criteria for vehicle formations. The interconnection between vehicles (i.e., which vehicles are sensed by other vehicles) is modeled as a graph, and the eigenvalues of the Laplacian matrix of the graph are used in stating a Nyquist-like stability criterion. The relationship between the location of the Laplacian eigenvalues and the graph structure is used to identify desirable and undesirable formation interconnection topologies.

Keywords: decentralized control, graph theory, stability criteria, vehicles

1. INTRODUCTION

Recent years have seen the emergence of control of vehicle formations as a topic of significant interest to the controls community. Applications of this span a wide range, including mobile robotics, traffic control, satellite clusters and UAV formations. A recent study (Air Force Scientific Advisory Board, 1995) identified this area as needing fundamentally new control paradigms.

Broadly speaking, this problem falls within the domain of decentralized control, but it possesses several unique aspects. The first is that vehicles in a formation are, as a rule, *dynamically decoupled*, meaning the motion of one vehicle does not directly affect any of the other vehicles. Instead, the vehicles are coupled through the *task* which they are jointly asked to accomplish. Tasks of this nature include requiring a formation to fly in a specific pattern, distribute itself evenly over a specified area, or arrive simultaneously at specified endpoints. Other tasks include the assignment of roles to individual vehicles within a formation which enable the entire formation to accomplish a higher-level task. When the formation is dynamically coupled, that coupling constrains, or at least naturally suggests, what information must be available to each component of the decentralized controller. In the case of cooperative vehicle control, no such architecture is necessarily suggested. As such, a second unique aspect of cooperative vehicle control is the fact that the interconnection structure between vehicles is not a given. It may be available as a design parameter, or the control architecture may require sufficient flexibility to handle changes in the interconnection structure. By "interconnection," we mean the flow of information from one vehicle to another.

In this paper, we will focus on interconnections generated by the ability of one vehicle to sense another vehicle. Specifically, we will address the

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impact of the interconnection topology on the formation. As a rule, no vehicle will be able to sense the entire formation, rendering centralized control infeasible. Additionally, the interconnection topology is itself dynamic, in that the ability of a vehicle to sense another vehicle can change due to outside influences or to changes in the formation itself. As such, a control law which is optimized for one topology may exhibit poor performance, or even instability, for another topology.

A natural way to model the interconnection topology is as a graph. Each vehicle is modeled as a node on the graph, and an arc joins node i to node j if vehicle j is receiving information from vehicle i. To accommodate the full range of possible topologies, we will consider directed graphs, meaning bidirectional communication is not assumed. Several authors (Desai *et al.*, 2001; Mesbahi and Hadaegh, 2001) have used graph-theoretic ideas in control of vehicle formations. In both those papers, the formation implements a variant of a leader-follower architecture. In graph-theoretic terminology, such a formation is *acyclic*, meaning no sequence of arcs leads from a node back to itself.

In this paper, we consider graphs which contain cycles and which therefore avoid the disturbance rejection problems associated with leader-follower architectures (Yanakiev and Kanellakopoulos, 1996) and are more robust to loss of individual links. A key challenge for formations of this sort is stability analysis, because cycles in the graph introduce a global component to each vehicle's dynamics which depends on both the structure of the graph and the vehicle dynamics.

Our goal in this paper is to derive stability criteria for the formation which facilitate controller design on the local level. Central to this development will be the use of the Laplacian of the graph, a matrix representation of the graph whose spectral properties can be related to structural properties of the graph (Chung, 1997; Merris, 1994). The Laplacian has been used previously in the study of chaos in interconnected oscillators (Heagy *et al.*, 1994; Pecora and Carroll, 1998). In this paper, we take a control-theoretic approach to stability analysis of interconnected vehicles. For the problem of relative formation stabilization, we will present a Nyquist-like criterion for formation stabilization, and we use the spectral properties of the Laplacian to evaluate desirable structural properties of the graph. In a companion paper, (Fax and Murray, 2002), we will study techniques by which information can be shared between vehicles to improve stability margins and formation performance.

2. PROBLEM SETUP

The problem we consider is the stabilization of a set of vehicles where only relative measurements are available to any given vehicle. Problems of this type include vehicle platoons (Yanakiev and Kanellakopoulos, 1996) and satellite formations (Yeh and Sparks, 2000). We consider a set of N vehicles, whose (identical) linear dynamics are denoted

$$\dot{x}_i = Ax_i + Bu_i \tag{1}$$

$$y_i = C_1 x_i \tag{2}$$

$$z_{ij} = C_2(x_i - x_j),\ j \in J_i \tag{3}$$

where $i \in [1, N]$ is the index for the vehicles in the flock. Note that each vehicle's dynamics are decoupled from the vehicles around it. The measurement y_i represents absolute state measurements, and z_{ij} represents relative state measurements. We will assume henceforth that no absolute state measurements exist, or that an inner loop has already been closed around them. The set $J_i \subset [1, N]\backslash\{i\}$ represents the set of vehicles which vehicle i can sense. It is assumed that $J_i \neq \emptyset$, meaning each vehicle can see at least one other vehicle. Note that a single vehicle cannot drive all the z_{ij} signals to zero; the errors must be synthesized into a single signal. For simplicity, we will assume that all relative state measurements are weighted equally to form one relative measurement:

$$z_i = \frac{1}{|J_i|} \sum_{j \in J_i} z_{ij}. \tag{4}$$

Of course, we could also weight different measurements differently. We also define a decentralized control law $K(s)$ which maps y_i, z_i to u_i, represented in state-space form by

$$\dot{v}_i = Rv_i + Sz_i \tag{5}$$

$$u_i = Tv_i + Vz_i. \tag{6}$$

We now consider the system of all N vehicles together. Let a hatted matrix, for example $\widehat{A}$, represent the matrix A repeated N times along the diagonal. Using this notation, we can represent the system of N vehicles as

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \widehat{A} + \widehat{BVC_2} L_{(n)} & \widehat{BT} \\ \widehat{SC_2} L_{(n)} & \widehat{F} \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix}. \tag{7}$$

where $L_{(n)}$ is defined in the following way. Let L be the $N \times N$ matrix defined by

$$L_{ii} = 1 \tag{8}$$

$$L_{ij} = \begin{cases} -\dfrac{1}{|J_i|}, & j \in J_i \\ 0, & j \notin J_i. \end{cases} \tag{9}$$

Letting n be the dimension of x_i, $L_{(n)}$ is an $Nn \times Nn$ matrix and is defined by replacing each element of L with that element multiplied by I_n, thus generating a version of L dimensionally compatible with x_i. The resulting system is block diagonal with the exception of $L_{(n)}$.

3. LAPLACIANS OF GRAPHS

Before we consider the graph-theoretic aspects of L, we introduce some terminology. There are many introductory texts on graph theory; Diestel (1997) is one. A *directed graph* $\mathcal{G}$ consists of a set of vertices, or nodes, denoted $\mathcal{V}$, and a set of arcs $\mathcal{A}$, where $a = (v, w) \in \mathcal{A}$ and $v, w \in \mathcal{V}$. The first element of a is denoted $\text{tail}(a)$, and the second is denoted the $\text{head}(a)$. It is said that a points from v to w. We will assume that $\text{tail}(a) \neq \text{head}(a)$, meaning that the graph has no loops. We also assume that each element of $\mathcal{A}$ is unique. A graph with the property that $(v, w) \in \mathcal{A} \Rightarrow (w, v) \in \mathcal{A}$ is said to be *undirected*, and the pair of arcs can be modeled as a single edge with no direction associated to it. The *in-degree* of a vertex v is the number of arcs with v as its head, and the *out-degree* is the number of arcs with v as its tail.

A *path* on $\mathcal{G}$ of length N from v_0 to v_N is an ordered set of distinct vertices $\{v_0, v_i, \ldots, v_N\}$ such that $(v_{i-1}, v_i) \in \mathcal{A}\ \forall i \in [1, N]$. An *N-cycle* on $\mathcal{G}$ is defined the same as a path except that $v_0 = v_N$, meaning the path rejoins itself. A graph without cycles is said to be acyclic. A graph with the property that the set of all cycle lengths has a common divisor M other than one is said to be *M-periodic*. A graph with the property that for any $v, w \in \mathcal{V}$, there exists a path from v to w, is said to be *strongly connected*.

We now turn to matrices associated with graphs. For this purpose, we assume that the vertices of $\mathcal{G}$ are enumerated, and each is denoted v_i. The adjacency matrix of a graph, denoted A, is a square matrix of size $|\mathcal{V}|$, defined by $A_{ij} = 1$ if $(v_j, v_i) \in \mathcal{A}$, and is zero otherwise. Note that A uniquely specifies a graph. Let D be the matrix with the in-degree of each vertex along the diagonal (assume each vertex has nonzero in-degree). The Laplacian of the graph is defined as [2]

$$L = I - D^{-1}A. \tag{10}$$

For the graph shown in Figure 1, the arc set (with the dashed arc omitted) is

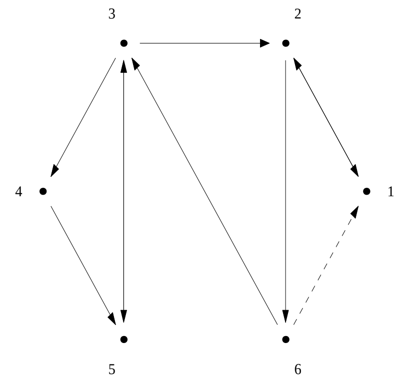

Fig. 1. Sample Directed Graph

$$\mathcal{A} = \{(1,2),(2,1),(2,6),(3,2),(3,4),\\(3,5),(4,5),(5,3),(6,3)\} \tag{11}$$

and the Laplacian is

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -0.5 & 1 & -0.5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -0.5 & -0.5 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -0.5 & -0.5 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{12}$$

The Laplacian matrix is an object of study within graph theory. Specifically, the eigenvalues of the Laplacian can be related to structural properties of its graph. We now note some basic, well-known properties of the Laplacian (Chung, 1997; Merris, 1994):

(1) $0 \in \lambda(L)$, and the associated eigenvector is $1_{N\times 1}$. If $\mathcal{G}$ is strongly connected, then the zero eigenvalue is simple.
(2) All eigenvalues of L are located in a disk of radius 1 in the complex plane centered at $1 + j0$. This can be shown by applying Gershgorin's theorem to the rows of L.
(3) If $\mathcal{G}$ is aperiodic, then no eigenvalues (other than the zero eigenvalue) will lie on the boundary of the Gershgorin disk. if $\mathcal{G}$ is M-periodic, then L has M eigenvalues on the boundary with an angular spacing of $\frac{2\pi}{M}$.
(4) If $\mathcal{G}$ is an undirected graph, all eigenvalues of L are real.

Most of these results can be deduced by observing that $D^{-1}A$ is nonnegative and applying concepts from Perron-Frobenius theory (Horn and Johnson, 1985). In the sections which follow, we will identify the role Laplacians play in formation stability analysis, and use the ideas mentioned above to evaluate the effects of certain formation interconnection structures on formation stability.

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[2] Some references define L as $D - A$. Also, some references use the out-degree in the definition of the Laplacian for directed graphs. The two are essentially equivalent, and our application is better suited to use of the in-degree.

4. STABILIZATION OF VEHICLE FORMATIONS

We wish to consider the relationship between graph Laplacians and formation stabilization. We show the following to be true:

Theorem 1. A local controller $K(s)$ stabilizes the formation dynamics in Equation (7) iff it simultaneously stabilizes the set of N systems

$$\begin{aligned} \dot{x} &= Ax + Bu \\ z &= \lambda_i C_2 x \end{aligned} \tag{13}$$

where λ_i are the eigenvalues of L.

PROOF. We will show the above to be true by transforming the closed-loop dynamics in the following way: Let T be a Schur transformation of L, meaning the unitary matrix such that $U = T^{-1}LT$ is upper triangular with the eigenvalues of L along the diagonal (Horn and Johnson, 1985). Clearly, $T_{(n)}$ is a Schur transformation of $L_{(n)}$. This transformation has the following useful property, a clear consequence of the block structure of the relevant matrices:

Lemma 2. Let X be an $r \times s$ matrix, and Y be an $N \times N$ matrix. Then

$$\widehat{X} Y_{(s)} = Y_{(r)} \widehat{X}. \tag{14}$$

Applying this property to the system dynamics, we see if we let $\tilde{x} = T_{(n)}x$, and $\tilde{v} = T_{(m)}v$, we can rewrite the system equations as

$$\begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{v}} \end{pmatrix} = \begin{pmatrix} \widehat{A} + \widehat{BJ_2C_2}U_{(n)} & \widehat{BH} \\ \widehat{G_2C_2}U_{(n)} & \widehat{F} \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{v} \end{pmatrix}.$$

The elements of the transformed system matrix are block diagonal with the exception of upper triangular $U_{(n)}$. This means that stability of this system is equivalent to the stability of the systems along the diagonal, i.e.

$$\dot{\tilde{x}}_i = (A + BJ_1C_1 + \lambda_i J_2 C_2)\tilde{x}_i + BJ_2C_2\tilde{v}_i \tag{15}$$

$$\dot{\tilde{v}}_i = (G_1C_1 + \lambda_i G_2 C_2)\tilde{x} + F\tilde{v} \tag{16}$$

which is equivalent to the controller $K(s)$ stabilizing the system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ z &= \lambda_i C_2 x \end{aligned} \tag{17}$$

In this context, the zero eigenvalue of L can be interpreted as the unobservability of absolute motion of the formation in the measurements z_i. It seems that a prudent controller design strategy is to close an inner loop around y_i such that the result system is stable, and then to close an outer loop around z_i which achieves desired formation performance. For the remainder of this paper, we concern ourselves solely with the outer loop. Hence, we assume from now on that C_1 is empty and that A has no eigenvalues in the open RHP. We do not wish to exclude eigenvalues along the $j\omega$ axis because many vehicle formations (e.g. vehicle platoons, satellite clusters) possess those, and the presence of unobservable secular or periodic motion of the formation may be tolerable in those cases. If $K(s)$ stabilizes the system in Equation (17) for all λ_i other than the zero eigenvalue, we will say that it stabilizes the relative formation dynamics.

In general, proving simultaneous stabilization results can be difficult. This set of systems is special, in that they differ only by a complex scalar. For single-input,single-output (SISO) systems, we can state a second version of Theorem 1 which is useful for stability and robustness analysis:

Theorem 3. Suppose $G(s) = C_2(sI - A)^{-1}B$ is a SISO system. Then $K(s)$ stabilizes the relative formation dynamics iff the net encirclement of $-\lambda_i^{-1}$ by the Nyquist plot of $K(s)G(s)$ is zero for all nonzero λ_i.

PROOF. The Nyquist Criterion states that stability of the closed loop system in Theorem 1 is equivalent to the number of CCW encirclements of $-1 + j0$ by the forward loop $\lambda_i G(j\omega)K(j\omega)$ being equal to the number of RHP poles of $G(s)$, which is assumed to be zero. This criterion is equivalent to the number of encirclements of of $-\lambda_i^{-1}$ by $G(j\omega)K(j\omega)$ being zero.

Note that because the vehicle is likely to have poles on the $j\omega$ axis, care must be taken when interpreting the Nyquist plot.

In the case where $G(s)$ is a multi-input, multi-output (MIMO) system, the formation can be thought of as a structured uncertainty of the type scalar times identity (Zhou and Doyle, 1998), where the scalars are the Laplacian eigenvalues. More specifically, we shall write the eigenvalues as $\lambda_i = 1 + \mu_i$ and consider bounds on μ_i. Suppose it is known that $|\mu_i| \leq M$ for all nonzero λ_i. If we close the loop around the unity block and leave $\mu_i I$ as an uncertainty, the resulting lower block is $C(s) = G(s)K(s)(I + G(s)K(s))^{-1}$, which is assumed to be stable. The following result from robust control theory then applies:

Theorem 4. $K(s)$ stabilizes the relative formation dynamics of the MIMO formation $G(s)$ if

$$\rho(C(j\omega)) < M^{-1} \;\; \forall \omega \in (-\infty, \infty) \tag{18}$$

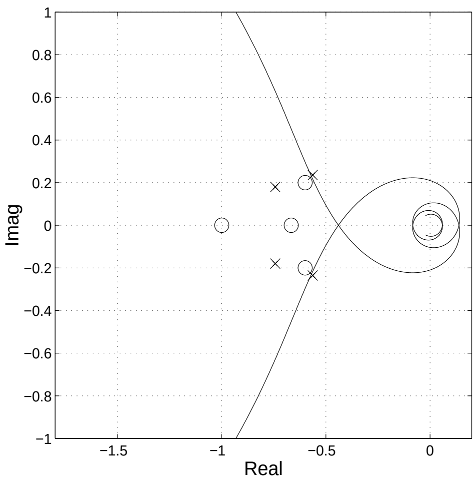

Fig. 2. Formation Nyquist Plot

Let us consider an example. Let $G(s) = \frac{e^{-sT}}{s^2}$ and $K(s) = K_p + K_d s$. This corresponds to a double integrator with a time delay being controller by a PD controller. Figure 2 shows a formation graph and the Nyquist plot of $K(s)G(s)$ with the Laplacian eigenvalues. The black 'o' locations in Figure 2 correspond to the eigenvalues of the graph defined by the black arcs in Figure 2, and the 'x' locations are for eigenvalues of the graph when the dashed arc is included as well. The Nyquist plot relative to -1 reveals a system with reasonable stability margins — about 8 dB and 45 degrees. When one accounts for the effects of the formation, however, one sees that for the 'o' formation, the stability margins are substantially degraded, and for the 'x' formation, the system is in fact unstable. Interestingly, the formation is rendered unstable when additional information (its position relative to vehicle 6) is used by vehicle 1. We shall return to this point shortly.

5. LOCATION OF LAPLACIAN EIGENVALUES

The location of Laplacian eigenvalues has emerged as the parameter which enables formation stability to be analyzed on the local level. A natural question to ask is: how does formation structure affect eigenvalue placement? We begin by considering simple formation structures and their eigenvalue placement.

(1) Complete graph. The complete graph is one where every possible arc exists. In this case, the eigenvalues of a graph with N vertices are zero and $1 + \frac{1}{N-1}$, the latter repeated $N-1$ times. For large N, stabilization of the complete graph is equivalent to stabilizing an individual vehicle.
(2) Acyclic (directed) graph. This graph has all eigenvalues at $\lambda = 1$. This can be seen from the fact that the vertices can be ordered such that L is upper triangular. This is the "leader-follower" architecture discussed in the introduction. In this case, stabilization of the formation is equivalent to stabilizing a single vehicle, since the Nyquist criterion does not change. This is consistent with the notion that in a leader-follower architecture, the motion of the leader can be treated as a disturbance on the follower.
(3) Two-periodic undirected graph. A graph of this type would include a vehicle platoon with bi-directional position measurement. This graph will have an eigenvalue at 2, due to its periodicity, and all other eigenvalues will be real.
(4) Single directed cycle. This graph has eigenvalues at $1 - e^{j(i-1)/2\pi},\ i \in [1, N]$.

Figure 3 shows various eigenvalue regions for $-L$ and the corresponding regions for $-L^{-1}$. The region bounded by the solid line is the Gershgorin disc in which all eigenvalues must lie. Its inverse is the LHP shifted by -0.5. The dashed region is a bound in the magnitude of the nonzero eigenvalues of L. It corresponds to a shifted circle on the right hand side of Figure. Finally, the dash-dot line corresponds to a bound on the real component of the eigenvalues. The inverse of this bound corresponds to a circle which touches the origin.

If we consider the complete graph and the single directed cycle graph as representing two extremes — one with all eigenvalues at a single location, the other with eigenvalues maximally dispersed, we see that eigenvalue placement can be related to the rate of mixing of information through the network. When the graph is highly connected, the global component of an individual vehicle's dynamics are rapidly averaged out through the rest of the graph, and thus has only a minor effect on stability. When the graph is periodic, the global component of the dynamics introduces periodic forcing of the vehicle, and the rest of the network never averages it out. This is represented on the Nyquist plot by putting the inverse eigenvalues nearer to the imaginary axis, thus degrading stability margins.

We see that aperiodicity is a desirable property of formation interconnection topologies. With this insight, we can see why the system in Figure 2 loses stability margin when a link is added. The "solid" graph possesses two 3-cycles and two 2-cycles. When the dashed link is added, an additional 3-cycle is created, rendering the graph more nearly 3-periodic. This drives two of the eigenvalues nearer to the positions they would occupy if the graph were truly periodic, i.e., the -0.5 vertical.

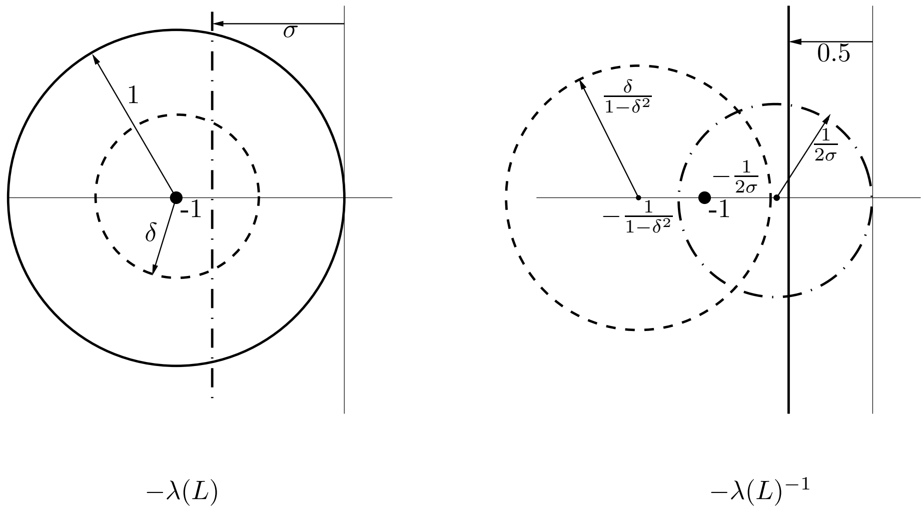

Fig. 3. Laplacian Eigenvalue Regions

6. CONCLUSIONS

In this paper, we have presented a Nyquist-like criterion for assessing the effects of the interconnection topology on the stability of a formation of vehicles. The criterion is local, in that it is stated in terms of the dynamics of a single vehicle, and it gives insight into the effects of graph structure on stability. To be sure, many variants to the problem presented here could be presented (e.g. weakly connected graphs, mixed relative and absolute position objectives). Our goal in this paper is to motivate the role graph-theoretic ideas can have in formation controller analysis and design.

We have focused thus far on the flow of sensed information through the network. A natural topic of interest is how stability (as well as disturbance rejection and other measures of interest) can be improved through transmission of information between vehicles. In the extreme case, vehicles could share all information (assuming strong connectivity of the transmitted information graph), and each vehicle could essentially realize a centralized control law. However, this approach has obvious drawbacks in terms of bandwidth and computational complexity. In a companion paper (Fax and Murray, 2002), we will present strategies for sharing minimal amounts of information between vehicles, and how that information can be used to render the formation more robust to changes in the various topologies.

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