Information Flow and Cooperative Control of Vehicle Formations

J. Alexander Fax   Richard M. Murray

Engineering and Applied Science
California Institute of Technology
Pasadena, CA 91125
{fax,murray}@cds.caltech.edu

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Abstract

Vehicles in formation often lack global information regarding the state of all the vehicles, a deficiency which can lead to instability and poor performance. In this paper, we demonstrate how exchange of minimal amounts of information between vehicles can be designed to realize a dynamical system which supplies each vehicle with a shared reference trajectory. When the information flow law is placed in the control loop, a separation principle is proven which guarantees stability of the formation and convergence of the information flow law regardless of the information flow topology.

1 Introduction

Recent years have seen the emergence of control of vehicle formations as a topic of significant interest to the controls community. Applications span a wide range, including mobile robotics, traffic control, satellite clusters and UAV formations. A recent study [1] identified this area as needing fundamentally new control paradigms.

Central to a discussion of cooperative vehicle control is a determination of the nature of the *information flow* throughout the formation. We will distinguish between two types of information flow: *sensed information*, meaning the ability of a single vehicle to sense some information (e.g. relative position) about another vehicle in a way which involves no action on the part of that vehicle, and *transmitted information*, meaning transfer of information between two vehicles which requires some action on the part of both the transmitter and recipient of that information. Sensing and transmission are the means by which each vehicle acquires the information necessary to perform its task within the formation. The sensing and transmitted information topologies are themselves dynamic, meaning they are subject to change due to changes in the formation itself or due to outside influences.

In [4], we considered the effect of the sensing network topology on stability of a vehicle formation. The sensing paths were modeled as a graph, and stability concerns arose when cycles were present in the graph. We proved a Nyquist-like criterion which used the eigenvalues of the Laplacian matrix of the graph in determining stability of the formation. In that paper, no information exchange between vehicles was considered. In this paper, we extend our other work by focusing on improving performance through the exchange of information between vehicles. From a performance perspective, sparseness in the sensing graph means that each vehicle has only a limited picture of the behavior of the formation, and cannot therefore make informed control decisions. We will demonstrate an information flow law which realizes a discrete-time dynamic system whose convergence enables the vehicles to achieve *consensus* regarding the formation center. Our method is driven by the need to ensure stability of the formation and robustness to changes in the information flow topology. As such, our methodology differs from traditional approaches to achieving consensus [7].

2 A Motivating Example

In [4], we discussed how sparseness in the sensing graph can lead to poor stability margins. In this chapter, we begin our discussion of information flow design by examining the effect of sparseness on performance. Returning to our example in that paper, suppose six vehicles whose dynamics are double integrators in the plane are asked to take up positions on the points of a regular hexagon relative to one another. Figure 1 shows the trajectories followed by the six vehicles as they approach their target positions. The initial positions are marked with an 'o,' and the final positions are marked with an 'x.' While the formation is stable and achieves its desired position (as verified in the previous chapter), the trajectories followed by the vehicles are very circuitous and far from optimal. This is a clear consequence of the fact that individual vehicles do not have global knowledge of the behavior of the formation. Specifically, the vehicles do not have knowledge of the formation center, which would enable them to position themselves relative to the formation center rather than relative to the small number of vehicles visible to each.

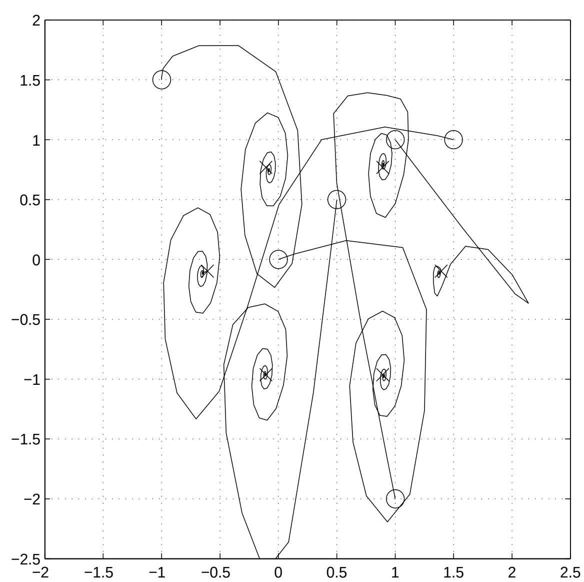

Figure 1: Hexagon Acquisition, No Information Flow

In this chapter, we will explore strategies for flow of information which enable the vehicles to acquire the information they lack. In doing so, we must keep in mind several areas of concern. The first is the interaction of the information flow loops with the vehicle dynamics and the possible consequences for stability. The tools for formation stability analysis developed in the previous chapter will be useful in this regard. The second is the need of the information flow policy to be robust to changes in the graph which defines the flow of transmitted information, as well as lack of knowledge at the local level of global structures in the graph. Finally, we wish to pursue information flow policies which minimize the amount of information exchanged between vehicles, and to avoid information flow laws which essentially enable each vehicle to perform a centralized control law computation, for reasons outlined in the introduction.

There is no limit to the complexity which information exchange policies could exhibit. Because our interest is in the interaction of the information exchange with vehicle dynamics, we will eschew the approach to consensus among distributed systems found in references such as Lynch [7], and instead pursue a "bottom-up" approach, in which we begin with the simplest forms of information exchange which remain amenable to stability analysis. As we shall demonstrate, even simple information exchange, when properly designed, can yield significant results. Hopefully, the methodology used in this and the previous chapter contains elements which can be applied to analyzing the interaction of more complex information exchange protocols with dynamical systems.

3 An Information Flow Paradigm

3.1 Problem Setup

In the previous chapter, we assumed that sensed information was available instantaneously, and we used a continuous-time model of the vehicle dynamics. In this paper, we will assume that information takes a fixed time T to travel between vehicles. To facilitate analysis, we also model our vehicle as a discrete time dynamical system:

$$\begin{array}{rcl} x^i_{k+1} & = & P_A x^i_k + P_B u^i_k \\ y^i_k & = & P_C x^i_k + P_D u^i_k \end{array} \tag{1}$$

where k is the time step of duration T and i is the vehicle index. As in [4], the error signal used by each vehicle is

$$z^i_k = \frac{1}{|J^S_i|} \sum_{j \in J^S_i} y^i_k - y^j_k, \tag{2}$$

meaning an average of the relative error measurement available to each vehicle. The (presumed non-empty) index set J^S_i represents the set of vehicles visible to vehicle i, and we can form a directed graph based on those sets.[1] When Equation (2) is represented as a matrix, it takes the form

$$z_k = L_{(n)} y_k, \tag{3}$$

where L is the Laplacian of the graph, and the (n) subscript indicates that each element of L is replaced with I_n for dimensional compatibility. The Laplacian is defined as $I - D^{-1}A$, where A is the adjacency matrix of the graph, and D has the in-degree of each vertex along the diagonal [3]. In the future, the subscripts will be omitted, and dimensional compatibility will be assumed. The eigenvalues of L were shown to be useful in proving stability of the formation and in evaluating fitness of different formations. Note that the stability results of [4] can be reproduced for discrete time systems if one uses the discrete Nyquist criterion rather than the continuous one.

[1] In [4] we omitted the superscript S, which we include here to identify it as the sensed information index set, as opposed to the transmitted information index set.

Broadly speaking, any information flow consists of vehicles receiving a transmission from other vehicles and performing some computation using that information, information from previous transmissions, and sensed information. Each vehicle then transmits the results of their computation to other vehicles. We can view this process as a discrete-time dynamical system where the states are the information at each vehicle. Thus, a generic information flow law can be represented

$$p^i_{k+1} = f(p^i_k, \ldots, p^i_{k-N_1}, \{p^j_k, \ldots, p^j_{k-N_2} | j \in J^T_i\}, z^i_{k+1}, \ldots, z^i_{k-N_3}), \tag{4}$$

where J^T_i is the set which determines the transmitted information topology, and N_i indicate the number of previous time steps used for each information source. We will assume that $J^T_i = J^S_i$ for all i, meaning the sensed information graph and transmitted information graph are identical. We will therefore omit the superscript.

The information flow law we are going to investigate will mimic the structure of the sensed information, taking the following form.

$$p^i_{k+1} = \sum_{j \in J_i} p^j_k + (y^i_k - y^j_k) \tag{5}$$

or, in vector form:

$$p_{k+1} = G_{(m)} p_k + L_{(m)} y_k \tag{6}$$

where $G_{(m)}$ and $L_{(m)}$ are the directed adjacency matrix and Laplacian of the graph, dimensioned compatibly with the measurement vector y^i_k whose dimension is denoted m. Henceforth, we shall assume that $m = 1$, and dispense with the extra notation. For the information flow laws to be derived, one can replicate all the results by replacing the given transfer functions with the same transfer function repeated m times along the diagonal.

3.2 Convergence of the Information Flow Loop

We now analyze stability and convergence properties of the information flow law, making use of ideas from Perron-Frobenius theory. See [2, 6, 10] for a complete treatment of Perron Reobenius theory. As discussed in detail in [4], G is a nonnegative matrix whose Perron root is 1 and whose eigenvalues must lie in the unit circle. The graph is termed *strongly connected* if any two nodes can be joined by a path, and *aperiodic* if the lengths of cycles in the graph do not have a greatest common divisor other than 1. We assume the graph to be strongly connected[2], which implies that G has positive left and right Perron eigenvectors e_l, e_r. Define $E = e_r e_l^T$, where e_l, e_r are chosen such that $e_r^T e_l = 1$. We will make use of the following two results: (See [6], p. 498, and recall that the Perron eigenvalue is 1.)

Lemma 1 $G^j = E + (G - E)^j$.

Lemma 2 *The eigenvalues of $G - E$ are the eigenvalues of G with the Perron eigenvalue replaced with a zero eigenvalue.*

We now state and prove the following theorem:

[2]This assumption can be relaxed if a path exists from any node to a single strongly connected subgraph.

Theorem 3 *Suppose the directed graph $\mathcal{G}(G)$ is strongly connected and aperiodic, and let the input y_k be fixed in time. The steady state value of the dynamical system in Equation (6), when $p_0 = 0$, is*

$$p_{ss}^i = y^i - \sum_{j=1}^{N} e_l^i y^i \tag{7}$$

where e_l^i is the ith element of the left Perron eigenvector of G, scaled so that $\sum e_l^i = 1$.

Proof. Consider the evolution of Equation (6):

$$p_k = G^k p_0 + \left(\sum_{j=0}^{k} G^j \right) Ly \tag{8}$$

We assume that $p_0 = 0$, and we wish to find

$$p_{ss} = \lim_{k \to \infty} p_k, \tag{9}$$

if such a limit exists.

Substituting into Equation (8) via Lemma 1, we have

$$p_k = \left(\sum_{j=0}^{k} E^j + (G - E)^j \right) Ly. \tag{10}$$

Recalling that $E = e_r e_l^T$,and that L shares eigenvectors with G, we see that e_r and e_l are the eigenvectors of L corresponding to the zero eigenvector. Therefore, $EL = e_r e_l^T L = e_r 0 = 0$, and we can rewrite p_k as

$$p_k = \left(\sum_{j=0}^{k} (G - E)^j \right) Ly. \tag{11}$$

Because G is assumed irreducible and aperiodic, all non-Perron eigenvalues of G have modulus strictly less than one. Therefore, by Lemma 2, we see that $\rho(G - E) < 1$. The infinite expansion of p_{ss} therefore converges ([6], p. 301) and can be written:

$$\begin{aligned}
p_{ss} &= \left(\sum_{j=0}^{\infty} (G - E)^j \right) Ly && (12)\\
&= (I - G + E)^{-1} Ly && (13)\\
&= (L + E)^{-1} Ly && (14)\\
&= (L + E)^{-1} (L + E - E) y && (15)\\
&= (I - (L + E)^{-1} E) y && (16)
\end{aligned}$$

Now $Le_r = 0$, and $Ee_r = (e_r e_l^T) e_r = e_r (e_l^T e_r) = e_r$, so $(L + E) e_r = e_r \Rightarrow (L + E)^{-1} e_r = e_r$, and the above equation can be rewritten

$$\begin{aligned}
p_{ss} &= (I - (L + E)^{-1} E) y && (17)\\
&= (I - (L + E)^{-1} e_r e_l^T) y && (18)\\
&= (I - e_r e_l^T) y && (19)\\
&= (I - E) y. && (20)
\end{aligned}$$

We now interpret the above equation. The eigenvector e_r is known to be 1^T. The eigenvector e_l is positive, and is scaled such that $\sum e_l^i = 1$. The columns of E are therefore constant, and the rows are each e_l^T. Therefore, Equation (20) can be written

$$p_{ss}^i = y^i - \sum_{j=0}^{N} e_l^i y^i. \tag{21}$$

■

The information flow loop therefore has the effect of having the formation track the formation center, where the center is defined according to a weighting given by the graph. In this architecture, the weighting cannot be chosen, though in principle it could be set by unevenly weighting the information when performing the averaging. However, this would require global knowledge of the graph, which is assumed not to be available.

3.3 Shaping the Information Flow

In the above section, we looked at the response of the information flow law to a constant input. Of course, the input to the information flow law need not be constant; it will reflect the dynamics of the formation. The designer may wish to know the response of the filter to different inputs, and to design it to be within certain tolerances over the range of expected inputs. With that in mind, we consider a more general form for the information flow filter:

$$\begin{array}{lcl} q_{k+1} & = & \sum_{j=0}^{R} a_j q_{k-j} + \sum_{j=0}^{R} b_j G q_{k-j} + L y_k \\ p_k & = & \sum_{j=0}^{R} c_j q_{k-j}. \end{array} \tag{22}$$

In this version, we are computing our current information using information from previous time steps as well as information received from other vehicles through a filter. This formulation can also be used to account for the presence of additional delays in data transmission. As in the previous case, we wish to determine the steady-state value of the filter for a constant input to understand the effects of the filter.

We begin by checking stability of the information flow law using the tools from [4]::

Theorem 4 *The system in Equation (22) is (neutrally) stable if the transfer function*

$$F(z) = \frac{\sum_{j=0}^{R} b_j z^{R-j}}{z^{R+1} - \sum_{j=0}^{R} (a_j + b_j) z^{R-j}} \tag{23}$$

is (neutrally) stable and its Nyquist plot avoids encirclement of the negative inverse of any of the nonzero eigenvalues of L.

Proof. We can take the z-transform of Equation (22), setting aside the input, and rewrite it as follows:

$$\begin{array}{lcl} zq(z) & = & \sum_{j=0}^{R} a_j z^{-j} q(z) + \sum_{j=0}^{R} b_j G z^{-j} q(z) \\ & = & \sum_{j=0}^{R} (a_j + b_j) z^{-i} q(z) - \sum_{j=0}^{R} b_j L z^{-j} q(z) \end{array}$$

or, if we collect terms not including L and multiply both sides by z^R,

$$q(z) = -\frac{\sum_{j=0}^{R} b_j z^{R-j}}{z^{R+1} - \sum_{j=0}^{R}(a_j + b_j) z^{R-j}} Lq(z). \tag{24}$$

The transfer function in the above equation is $\hat{F}(z)$, and this equation is equivalent to the lower loop shown in Figure 4. This block diagram has the same structure as the system of vehicle formations examined in [4], where is was shown that the stability of this system is given by the Nyquist criterion stated above. Because one set of eigenvalues of this system corresponds to the open-loop dynamics, this system can be at best neutrally stable if $F(z)$ is itself neutrally stable. ■

We now turn to the steady-state performance of the information flow law. We assume that $c_j = b_j$, which will ensure that the information flow law does not add gain to the loop, and which will be useful in the stability proofs of Section 4. Additionally, if $c_j = b_j$, then it is only necessary for each vehicle to transmit $\sum_{j=0}^{R} b_j q_{k-j}^i$ to its neighbors. We also assume that $F(z)$ has all poles on the interior of the unit circle with the possible exception of a simple pole at 1. Finally, we assume that the polynomial $\sum_{i=0}^{R} a_i z^{R-i}$ has roots in the interior of the unit circle.

Theorem 5 *If $F(z)$ stabilizes L in the sense of Theorem 4, and under the above assumptions,*

$$p_{ss} = c\left(I - cE - (1-c)\left(I - c(G-E)\right)^{-1} G\right) y \tag{25}$$

where $a = \sum_{j=0}^{R} a_j$, $b = \sum_{j=0}^{R} b_j$, and $c = \frac{b}{1-a}$

Proof. If we transform Equation (22) to the z-domain, it can be written:

$$zq(z) = \sum_{j=0}^{R} a_j z^{-j} q(z) + \sum_{j=0}^{R} b_j G z^{-j} q(z) + L\frac{z}{z-1} y \tag{26}$$

Note that the extra transfer function multiplying y is due to the assumption that y is constant. We recall the Final Value Theorem for discrete time systems [5]:

$$\lim_{k\to\infty} x(k) = \lim_{z\to 1}(z-1)X(z). \tag{27}$$

We can rewrite Equation (26) as

$$(z-1)q(z) = \left(zI_N - \sum_{j=0}^{R} a_j z^{-j} - \sum_{j=0}^{R} b_j G z^{-j}\right)^{-1} zLy. \tag{28}$$

The assumed neutral stability of the system implies that the matrix to be inverted is in fact nonsingular for all $|z| > 1$. We can therefore set $z = 1 + \epsilon$, and extract a z from the inverted matrix to cancel the z which postmultiplies it. The resulting expression is

$$(z-1)q(z) = \left(I_N - \sum_{j=0}^{R} a_j z^{-j-1} - \sum_{j=0}^{R} b_j G z^{-j-1}\right)^{-1} Ly. \tag{29}$$

The invertibility of the matrix implies we can again use the expansion of the inverse, Lemma 2 and the fact that $EL = 0$ to rewrite our expression as

$$(z-1)q(z) = \left[I_N - \sum_{r=1}^{\infty} \left(\sum_{j=0}^{R} a_j z^{-j-1} - \sum_{j=0}^{R} b_j (G-E) z^{-j-1} \right)^r \right] Ly. \tag{30}$$

We now examine the eigenvalues of our system with G replaced by $G - E$ as ϵ goes to zero. The eigenvalues of this system are the poles of $F(z)$ with its loop closed about each eigenvalue of L. By assumption, they are all stable with the possible exception of a pole at 1, corresponding to closing the loop about the zero eigenvalue of L. When we replace G with $G - E$, we leave the eigenvalues unchanged with the exception of the Perron eigenvalue, which is now zero. The corresponding eigenvalues of our system are the eigenvalues of Equation (30) with the $G - E$ term removed. These eigenvalues are the roots of $\sum_{j=0}^{R} a_j z^{R-j}$, which are in the interior of the unit disk by assumption. We therefore conclude that the series in Equation (30) converges even when $\epsilon = 0$, and represent the equation in inverted form:

$$\lim_{k\to\infty} q(k) = (I_N - a - b(G-E))^{-1} Ly \tag{31}$$

where $a = \sum_{j=0}^{R} a_j$ and $b = \sum_{j=0}^{R} b_j$, or

$$\lim_{k\to\infty} q(k) = \frac{1}{1-a} (I_N - c(G-E))^{-1} Ly \tag{32}$$

where $c = \frac{b}{1-a}$. When $c = 1$, the result is clearly the same as derived earlier. When $c \neq 1$, we proceed as follows:

$$\begin{aligned}
\lim_{k\to\infty} q(k) &= \frac{1}{1-a} (I_N - c(G-E))^{-1} Ly \\
&= \frac{1}{1-a} \Big((I_N - c(G-E))^{-1} (I_N - c(G-E) - (1-c)G - cE) \Big) y \\
&= \frac{1}{1-a} \Big(I_N - (I_N - c(G-E))^{-1} (1-c)G + cE \Big) y \\
&= \frac{1}{1-a} \Big(I_N - cE - (1-c) (I_N - c(G-E))^{-1} G \Big) y.
\end{aligned}$$

The output $p(z)$ is, (again via FVT)

$$\lim_{k\to\infty} p(k) = c \Big(I_N - cE - (1-c) (I_N - c(G-E))^{-1} G \Big) y. \tag{33}$$

■

Note that $c = 1$ corresponds to $a + b = 1$, which implies that the system has a pole at 1. When $c = 1$, we recover the steady-state result of Theorem 3, only we now see it to be true for any information flow filter with a simple pole at 1 (and which stabilizes the graph). When $c < 1$, the steady-state is offset by an additional term. Note that when $c = 1$, the vehicles all agree on the location of the formation center (expressed in each vehicle's coordinates), while when $c < 1$, they do not. We can say that when $c = 1$, the vehicles achieve *consensus* on formation center, whereas when $c \neq 1$ they do not. From this perspective, having $c = 1$ appears to be a desirable property of the information flow filter. However, when $c = 1$, the system is only neutrally stable. The reason the filter converges to a steady state is because the input passed through L, whose kernel is equal to the Perron eigenvector of G. However, the presence of noise or sensor errors has the potential to introduce drift. Additionally, the eigenvalue at 1 means that old information never decays out.

3.4 Example

To understand the effects of shaping the information flow, we will consider two examples. The first filter is in Equation (6). In this case, following Equation (23) $F_1(z) = \frac{1}{z-1}$. The second filter is given by

$$p_{k+1} = 1.0625p_k - 0.2313p_{k-1} + 0.1875Gp_k - 0.0188Gp_{k-1} \quad (34)$$

$$q_k = 0.1875p_k - 0.0188p_{k-1} \quad (35)$$

This corresponds to $F_2(z) = \frac{0.1875(z-0.1)}{(z-0.25)(z-1)}$. The pole at 1 means that $c = 1$ in both cases. Figure 2 shows the Nyquist plot for these two filters. The first lies along the -0.5 vertical. Points on that line correspond to periodic graphs, (see [4]) which confirms Theorem 3. The second lies entirely to the right of the -0.5 vertical, meaning that it will stabilize any graph. Figure 3 shows the response of the two filters to a step response for a sparsely connected graph. While both settle in approximately 0.5 sec (using a time step of 0.02 sec), the first filter exhibits ringing due to the proximity of the closed loop poles to the unit circle. The second filter has a much smoother response. We see how the information flow filter can be designed to achieve desirable responses and robustness to uncertainty in the graph. The information flow filter should also be designed to have good tracking properties over the frequency range of the vehicle dynamics.

4 Information Flow in the Loop

The information flow filter supplies each vehicle with the information it cannot sense: a formation center about which to do control. In this setting p is the input to the controller $K(z)$. A block diagram for this architecture is shown in Figure 4. As before, we can analyze stability with respect to uncertainties in the graph by isolating L and applying the Nyquist criterion as in [4]. In this case, one determines stability by analyzing the Nyquist plot of

$$F(z)(1 + K(z)P(z)). \quad (36)$$

For a given plant and controller, the information flow loop can be designed to provide desirable margins. However, care must be taken in interpreting the stability margins derived from this plot. The gain and phase margins of this plot do not correspond to uncertainties in the plant in the typical fashion due to the location of $P(z)$ in the transfer function. Instead, the correspond more directly to uncertainties in L. Small variations in $P(z)$ can produce unexpected perturbations of the Nyquist plot. A reasonable design methodology is to design $K(z)$ to stabilize $P(z)$, without regard to the formation (remember that stabilizing the formation is never easier than stabilizing an individual vehicle) and then design $F(z)$ to stabilize L. However, the coupling between the dynamics of the two can produce unexpected results. In the next section, we will explore a means to improve this situation.

4.1 Separation via Feedforward Correction

The information flow algorithm presented earlier is necessarily reactive; it does not anticipate the motion of the cluster. A logical means of improving performance is to supply the information flow loop with feedforward information regarding the expected motion of the formation. In this section, we shall explore augmentations to the information flow loop which follow this paradigm.

Recalling that the information represents an averaged position of the vehicles' positions, a logical choice for a feedforward signal is the anticipated change in vehicle position. This can be calculated

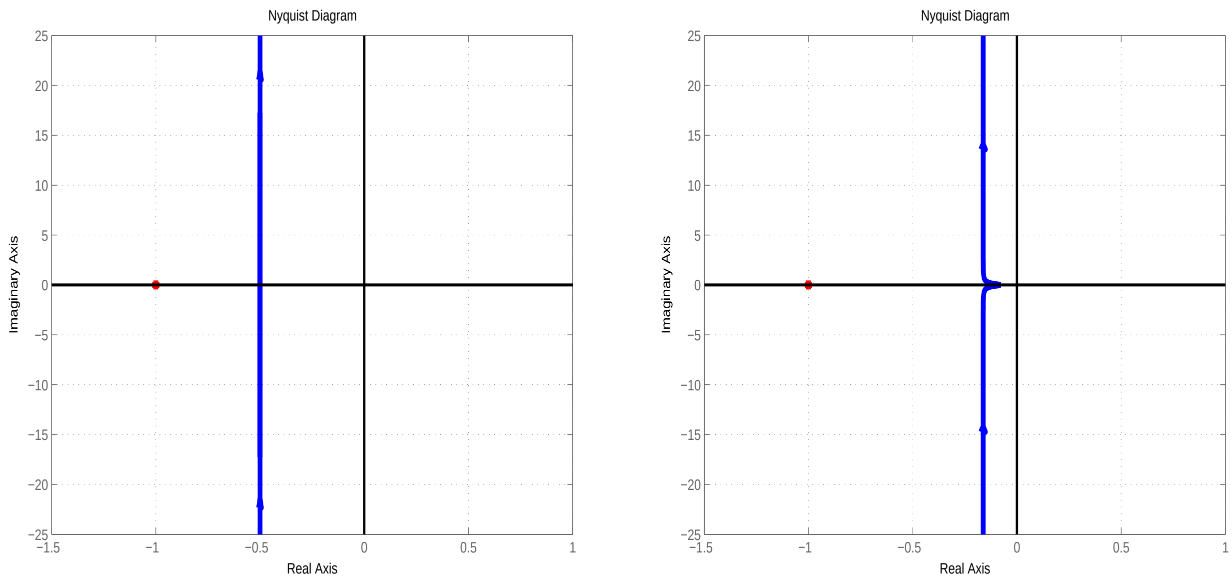

Figure 2: Information Filter Nyquist Plots

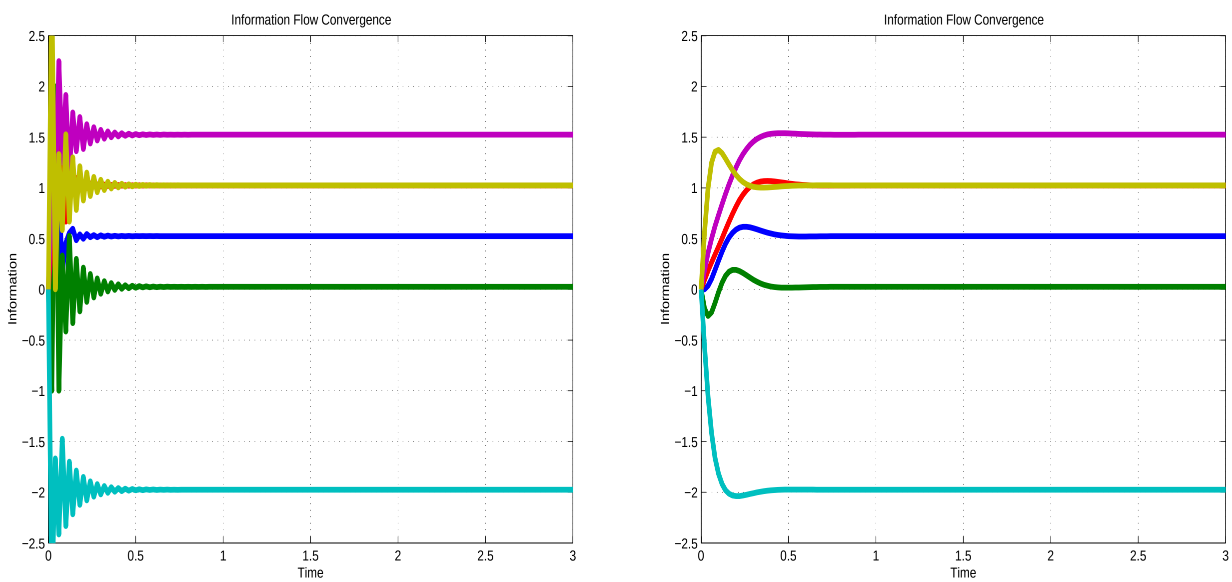

Figure 3: Information Filter Convergence

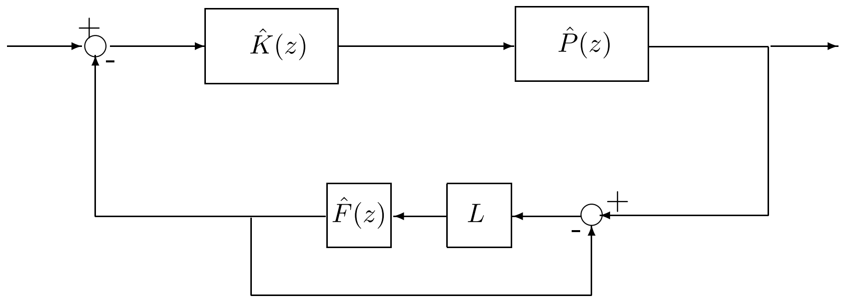

Figure 4: Block diagram of Information Flow in the Loop

by using each vehicles' control signal $u(z)$ as the input to a model of the plant, denoted $\tilde{P}(z)$, and differencing that. The resulting signal

$$w^i(z) = (1 - z^{-1})\tilde{P}(z)u^i(z) \tag{37}$$

is then transmitted in addition to the signal $q(z)$ and used by each vehicle as a correction term to p. For example, we would replace Equation (6) with the following information flow law:

$$\begin{aligned} q_{k+1} &= G(q_k + w_k) + Ly_k \quad (38)\\ p_k &= q_k + w_k \quad (39) \end{aligned}$$

In this case, the transmitted quantity is $p_k^i = q_k^i + w_k^i$, as is clear from is premultiplication by G. Of course, this feedforward correction term is only current if the control signal is delayed by a time step before application to the plant to allow a time step for the information to reach the other vehicles. Alternatively, each vehicle could delay the use of its sensed information until it receives the transmitted information from that vehicle.

To allow for information flow laws more general than Equation (37), we will let $w(z)$ take on the more general form

$$w(z) = H(z)\tilde{P}(z)u(z). \tag{40}$$

When $H(z)$ is chosen properly, the following result can be derived:

Theorem 6 *Choose $H(z)$ to be*

$$H(z) = \frac{1}{F(z) + 1}, \tag{41}$$

and suppose the feedback interconnection of $P(z)$ and $K(z)$ is well-posed. Then the relative formation dynamics are stabilized if and only if F(z) stabilizes L in the sense of Theorem 4 and $K(z)$ stabilizes $P(z)$.

Proof. By construction, $F(z)$ is biproper. Using our definition of $F(z)$, we can write $H(z)$ as

$$H(z) = \frac{z^{R+1} - \sum_{j=0}^{R}(a_j + b_j)z^{R-j}}{z^{R+1} - \sum_{j=0}^{R} a_i z^{R-j}}. \tag{42}$$

Note that $H(z)$ is stable due to the assumptions of Theorem 4.

We prove the presence of a separation principle for the system of equations, through the use of a transformation of coordinates which isolates the subsystems whose stability implies stability of the overall system. To to this, we first present the system of equations in state-space form. The state-space equations of motion for the plant are given in Equation (1). The predictor $\tilde{P}(z)$ is presumed to be identical to the plant $P(z)$, and has the same equations of motion with x, y replaced by $\tilde{x}, \tilde{y}$. The dynamics of the controller will be represented as

$$\begin{aligned} v_{k+1}^i &= K_A v_k^i + K_B p_k^i && (43) \\ u_k^i &= K_C v_k^i + K_D p_k^i. && (44) \end{aligned}$$

The information flow filter $F(z)$ is defined as found in Equation (22), but with the feedforward correction term added:

$$\begin{aligned} q_{k+1} &= \sum_{j=0}^{R} a_j q_{k-j} + G\left(\sum_{j=0}^{R} b_j G q_{k-j} + w_k\right) + L y_k && (45) \\ p_k &= \sum_{j=0}^{R} b_j q_{k-j} + w_k. && (46) \end{aligned}$$

Once again, it should be clear from the position of the quantity $\sum_{j=0}^{R} b_j G q_{k-j} + w_k$ Finally, the feedforward correction term is represented as

$$\begin{aligned} r_{k+1}^i &= \sum_{j}^{R} a_j r_{k-j}^i + \tilde{y}_k^i && (47) \\ w_k^i &= -\sum_{j}^{R} b_{k-j} r_k^i + \tilde{y}_k^i && (48) \end{aligned}$$

To simplify the representation of Equations (45,47) in state-space notation, we introduce the following notation. Let

$$H_A = \begin{pmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ a_R & a_{R-1} & \ldots & a_0 \end{pmatrix}, \qquad (49)$$

let

$$H_B = \begin{bmatrix} 0 & 0 & \ldots & 1 \end{bmatrix}^T, \qquad (50)$$

and let

$$H_C = \begin{bmatrix} b_R & \ldots & b_0 \end{bmatrix}, \qquad (51)$$

where H_B is dimensioned compatibly with H_A. For the information flow law and feedforward term, we use $\bar{q}^i$ to denote $[q_{k-R}^i, q_{k-R+1}^i, \ldots, q_k^i]^T$, and similarly for $\bar{r}$. The state-space representation of Equation (45) can thus be represented

$$\begin{aligned} \bar{q}_{k+1} &= \hat{H}_A \bar{q}_k + H_B G(H_C \bar{q}_k + w_k) + C L y_k && (52) \\ p_k &= H_C \bar{q}_k + w_k && (53) \end{aligned}$$

and of Equation (47) of

$$
\begin{aligned}
\bar{r}_{k+1} &= \hat{H}_A \bar{r}_k + \hat{H}_B \tilde{y}_k && (54)\\
w_k &= -\hat{H}_C \bar{r}_k + \tilde{y}_k. && (55)
\end{aligned}
$$

If one solves Equations (1,43,52,54) for the states, the resulting system of equations is

$$X_{k+1} = \Psi X_k, \tag{56}$$

where $X_k = [x_k^T, v_k^T, \tilde{x}_k^T, \bar{r}_k^T, \bar{p}_k^T]^T$ and

$$\Psi = \begin{pmatrix} P_A & P_B\Delta K_C & P_B K_D \Delta P_C & -P_B K_D \Delta H_C & P_B K_D \Delta H_C \\ 0 & K_A + K_B P_D \Delta K_C & K_B \Delta P_C & -K_B \Delta H_C & K_B \Delta H_C \\ 0 & P_B \Delta K_C & P_A + P_B K_D \Delta P_C & -P_B K_D \Delta H_C & P_B K_D \Delta H_C \\ 0 & C P_D \Delta K_C & H_B \Delta P_C & H_A - H_B P_D K_D \Delta H_C & H_B P_D K_D \Delta H_C \\ H_B L P_C & H_B P_D \Delta K_C & \phi P_C & -\phi H_C & H_A + \phi H_C \end{pmatrix}. \tag{57}$$

where $\phi = C(P_D K_D \Delta + G)$, and $\Delta = (I - P_D K_D)^{-1}$, which is invertible by assumption of well-posedness of the interconnection. If we apply the transformation

$$T = \begin{pmatrix} I & 0 & 0 & I & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & I & 0 & 0 & I \\ 0 & 0 & 0 & 0 & I \end{pmatrix} \tag{58}$$

to the system matrix we recover the matrix

$$T^{-1}\Psi T = \begin{pmatrix} \boxed{P_A} & 0 & 0 & 0 & 0 \\ -H_B L P_C & \boxed{H_A + H_B G H_C} & 0 & 0 & 0 \\ 0 & -K_B \Delta H_C & K_A + K_B P_D \Delta K_C & K_B \Delta P_C & 0 \\ 0 & -P_B K_D \Delta H_C & P_B \Delta K_c & P_A + P_B K_D \Delta K_C & 0 \\ H_B L P_C & -\phi H_C & H_B P_D \Delta K_C & H_B \Delta P_C & \boxed{H_A} \end{pmatrix}. \tag{59}$$

Stability of the system is equivalent to stability of the blocks along the diagonal. The first, P_A, is neutrally stable by assumption. The assumption that the information flow law stabilized the graph is equivalent to the second block, $H_A + H_B G H_C$, being stable. The third block along the diagonal, which comprises the third and fourth columns/rows, is stable if $K(z)$ stabilizes $P(z)$. (The reader will verify that this is the matrix derived when $K(z)$ and $P(z)$ are interconnected directly via feedback.) The final block, represents the states of $H(z)$, which is stable by the assumption in Theorem 4. We thus derive a separation principle for our formation which implies that design of the individual vehicle controller $K(z)$ and the information flow filter $F(z)$ can be decoupled. ■

Remarks Equation (59) can be interpreted in the following way. The first set of states are open-loop copies of the vehicles' dynamics, and represent mismatches in initial conditions between the predictor and the actual vehicle. The second set is identical to the dynamics of Equation (22), whose stability and convergence properties were studied above. The output of this set of states acts as a reference to N more copies of the vehicle dynamics in feedback interconnection with the local controllers, found in the third and fourth rows. We see, therefore, that the effect of this architecture is to supply the local controllers with a reference signal which, if implemented properly, represents the error of that vehicle relative to a common reference trajectory whose dynamics obey the open loop dynamics of an individual vehicle. The final set of states represent the feedfoward component. These states are unobservable in the motion of the vehicles, but are stable by design.

Several observations can be made regarding implementation. The first is that the motion of the formation is sensitive to mismatches between initial conditions of the vehicle and predictor. This can lead to drift of the cluster if the mismatch is in velocities. It may be possible to improve upon this through the use of an observer which will prevent the vehicle and predictor from diverging. This will be the subject of future investigations.

The second is that if $c \neq 1$, meaning the information flow loop does not converge to a common reference, then the vehicles' final positions will incorporate those errors as well (although the system is stable in this case as well). The position of the vehicles will also depend on the ability of the information flow filter to track the natural motion of the vehicles. When the natural motion of the vehicles is rest, we have seen that it achieves a proper steady state when $c = 1$. When the natural motion is secular drift or oscillation (corresponding to poles at the origin or along the $j\omega$ axis), the quality of the reference signal will depend on the ability of the information flow filter to track signals at the relevant frequencies.

We also note that the model of the plant $\tilde{P}(z)$ is not an observer, but a predictor of vehicle motion. The zero at 1 in $H(z)$ corresponds to differencing the input, which generally amplifies signal noise. However, the input to $H(z)$ is derived by integrating $u(z)$, so no net differencing takes place in the filter. In fact, it is possible to compress $P(z)$ and $H(z)$ into a single filter, but it is easier not to do so when proving stability.

Finally, we note that unlike the results of the previous chapter, this separation principle does *not* rely on the vehicles having identical plants or controllers. It merely relies on each vehicle's predictor matching the vehicle dynamics and on each vehicle implementing the same information flow and feedfoward correction computation. This eliminates a significant obstacle to implementation. A minor consequence is that when the vehicles have the same dynamics, the bulk motion of the formation itself obeys the dynamics of a single vehicle, while when the vehicles have different dynamics, that motion will be more complex.

4.2 Example

We return to the case with which we opened the chapter. If the information flow law together with feedforward compensation is enabled, the vehicles follow the trajectories shown in Figure 5. The trajectories are smoother, but still show some curving due to action of the control law prior to convergence of the information flow law. Figure 6 shows the trajectories followed by the vehicles when the information flow law is enabled one second prior to enabling the control loop. In this case, the vehicles follow straight lines to their targets. Note that the formation center is identical in the two cases despite the differing trajectories. This is due to the decoupling of the information flow law from expected formation motion.

5 Information Flow and String Stability

Thus far, we have seen that the proper design of information flow leads to improved stability due to the separation principle of Theorem 4 and improved vehicle trajectories due to the achievement of consensus among the vehicles as to the formation center. In this section, we turn our attention to disturbance rejection in the formation. A well-known area of concern within leader-follower formations is the possibility that a following vehicle can amplify disturbances of a leading vehicle. Depending on the length of the chain of vehicles, this can lead to unacceptably large disturbances

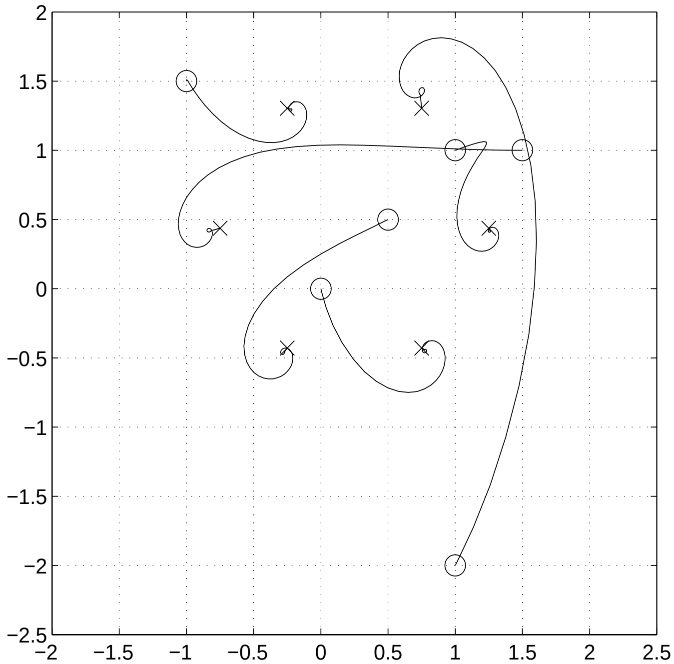

Figure 5: Hexagon Acquisition, Info Flow Enabled, no Info Pre-Convergence.

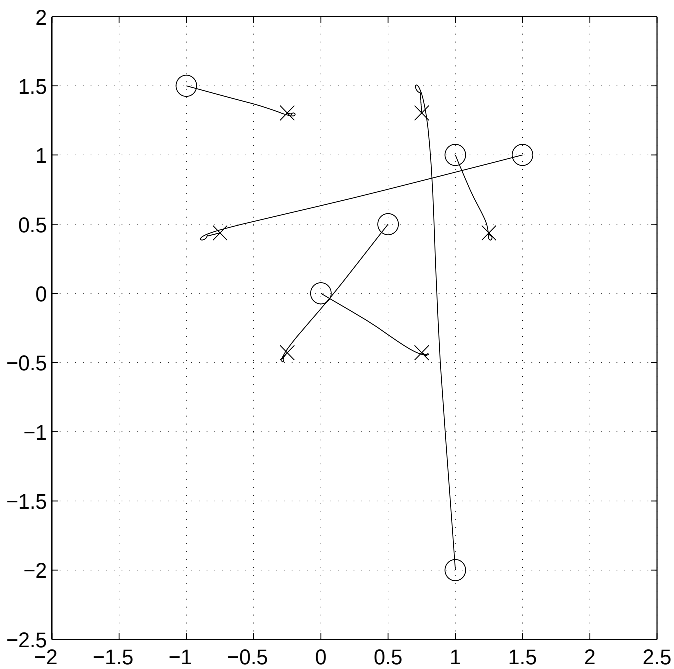

Figure 6: Hexagon Acquisition, Info Flow Enabled, with Info Pre-Convergence.

of the vehicles at the end of the chain. If we posit an infinite chain of vehicles, and define the difference between the positions of vehicles i and $i+1$ at time step k as $e^i(k) = y_k^{i+1}(t) - y_k^i(t)$, the effect of disturbance of the lead vehicle on the behavior of follower vehicles can be determined by looking at the sequence

$$\{\|e^1(k)\|, \|e^2(k)\|, \ldots, \|e^i(k)\|, \ldots\}. \tag{60}$$

If the sequence converges to zero for any bounded input disturbance to the lead vehicle, the formation is said to be *string stable*. If the sequence remains bounded, it is said to be *weakly string stable*. When the graph consists of a single chain, string stability is often determined by deriving the transfer function such that

$$e^{i+1}(z) = H(z)e^i(z) \tag{61}$$

and checking the infinity norm of $H(z)$. If it is less than one, then the sequence converges under the 2-norm, and thus the formation is string stable. If it is one, then the sequence remains bounded under the 2-norm, and the formation is weakly string stable. See [9] and the references therein for more formal definitions of string stability, and see [8] for a nonlinear approach.

A well-known property of leader-follower formation is that with only relative state knowledge of the immediately preceding vehicle, even weak string stability cannot be achieved [9, 11]. Candidate solutions explored in these references include having some knowledge of the lead vehicle's position or employing a variable speed spacing policy (i.e. having $h_{i,i+1} = h^0_{i,i+1} + h^1_{i,i+1}\dot{e}^i$.). The former appears to be the better strategy, but it relies on the ability to transmit the lead vehicle's position down the chain infinitely quickly.

With this in mind, we will analyze the string stability of a single chain formation with information flow and feedforward correction enabled. A complete analysis of this approach from the perspective of string stability is beyond the scope of this thesis, but insofar as the single chain is a known worst case, we rely on it to demonstrate the utility of the method. Because our system involves two pieces of information, namely position and transmitted information, we will need to consider the behavior of both as they are transmitted down the chain. We will denote relative position as

$$ye^i_k = y^{i+1}_k - y^i_k \tag{62}$$

and similarly for the other variables, e.g. pe^i_k representing the difference between p^{i+1}_k and p^i_k. From Equations (), we can derive the following:

$$\begin{aligned}
ye^i(z) &= P(z)ue^i(z) && (63)\\
\tilde{ye}^i(z) &= P(z)ue^i(z) && (64)\\
ue^i(z) &= K(z)pe^i(z) && (65)\\
pe^i(z) &= we^i(z) + (-H(z)+I)(pe^{i-1}(z) + ye^i(z) - ye^{i-1}(z)) && (66)\\
we^i(z) &= H(z)\tilde{ye}^i(z), && (67)
\end{aligned}$$

where

$$\begin{aligned}
P(z) &= P_C(zI - P_A)^{-1}P_B + P_D && (68)\\
K(z) &= K_C(zI - K_A)^{-1}K_B + K_D && (69)\\
H(z) &= -H_C(zI - H_A)^{-1}H_B + I. && (70)
\end{aligned}$$

These equations can be simplified to

$$\begin{aligned}
ye^i(z) &= P(z)K(z)\left[(I - H(z))\left(pe^{i-1}(z) - ye^{i-1}(z)\right) + ye^{-1}\right] && (71)\\
pe^i(z) &= (I - H(z))\left(pe^{i-1}(z) - ye^{i-1}(z)\right) + ye^{-1}, && (72)
\end{aligned}$$

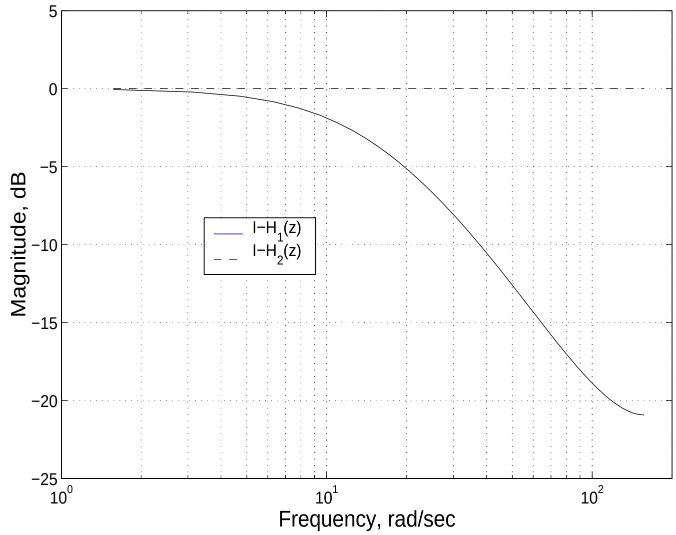

Figure 7: Information Flow Law Bode Plot, $I - H_1(z), I - H_2(z)$.

which can further be simplified to

$$ye^i(z) = (I - H(z))(pe^{i-1}(z) - ye^{i-1}(z) \quad (73)$$

$$pe^i(z) - ye^i(z) = (-H(z) + I)(pe^{i-1}(z) - ye^{i-1}(z), \quad (74)$$

where $C(z) = (I - P(z)K(z))^{-1} P(z)K(z)$. It is now clear that string stability of the quantity $pe^i(z) - ye^i(z)$ depends solely on the ∞-norm of $-H(z) + I$. Furthermore, the presumed stability of the closed loop transfer function $(I - P(z)K(z))^{-1} P(z)K(z)$ implies that if $pe^i_k - ye^i_k$ is bounded, then ye^i_k will be bounded, and if $pe^i_k - ye^i_k$ is decays to zero, then so will, ye^i_k. We thus arrive at the intriguing result that string stability is independent of the plant and controller, and depends solely on the design of the information flow filter. Recalling that $H(z)$ has a zero at $z = 1$, it follows that $I - H(z) = 1$ at $z = 1$, and thus the ∞-norm cannot be less than 1, so weak string stability is the best that can be achieved. However, design of information flow laws which achieve weak string stability does not appear difficult: Figure 7 shows the Bode plots of $I - H_1(z)$ and $I - H_2(z)$ as defined relative to $F_1(z),\ F_2(z)$ in Section 3.4. In both cases, the transfer functions achieve weak string stability. The first information flow law has $|I - H_1(z)| = 1$ at all frequencies, which does not imply particularly good disturbance rejection. The second has better string stability properties, realizing a gain of one only at DC.

Swaroop and Hedrick [8] argue that when lead vehicle information is not available, weak string stability is the best result than can be achieved, and these results do not contradict that. Additionally, because only weak string stability is achieved, it cannot be assumed that this condition is robust errors in the predictor model. However, it is significant that the information flow methodology presented here provides a systematic method for achieving the best possible (theoretical) result in a fashion which does not make assumptions on the plant or controller. While this result is only derived here for a single chain formation, because the result is essentially a consequence of separation principle, it should hold for more complex leader-follower architectures.

6 Conclusion

The information flow architecture presented in this chapter relies on two key ideas. The first is the use of dynamical systems as a paradigm for understanding information exchange between vehicles, and the design of a dynamical system which enables the vehicles to achieve consensus on the formation center. The second is the use of feedforward compensation to render the sensed and transmitted information timely. The resulting architecture achieves improvements in stability, vehicle trajectories, disturbance rejection, and robustness to changes in the interconnection structure. The architecture is flexible in that it does not rely on uniform vehicle dynamics, nor does it rely on a vehicle having any global knowledge of the information flow graph.

We have also seen limitations to the method in the course of the derivation. The first limitation is the need for an exact model of the vehicle dynamics. The sensitivity of the method to modeling errors has not been analyzed, nor has it been validated in an experimental setting. Simulation results do not expose high sensitivity to modeling errors. Another limitation is the sensitivity to mismatches in initial conditions, particularly in velocities, between the vehicle and the predictor. It may be possible to improve on this through the use of an observer rather than a predictor. A third limitation is the constraint that $c = 1$ in the information flow law. The need for consensus among vehicles forces the information flow law to be neutrally stable, which means that information never decays out. This renders the system sensitive to sensor errors that cause the vector of measurements Ly_k to have a component which lies along the Perron vector, leading to secular drift of the information flow. One possibilities for improving on this is having a protocol for resetting the information to zero periodically or in response to an event as a means of limiting any drift. Such a protocol could lie in a higher layer in the control architecture, and may itself require stability analysis. All these issues represent avenues of future research for improving an already potent method for vehicle formation control.

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