

On achievable accuracy for pose tracking

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“pose tracking” \triangleq recovering the incremental robot displacement given the output of a relative sensor

Central questions:

- How precise can an algorithm be, in principle?
- Do existing algorithms achieve this precision?

Background

- **Long-term goal:** characterizing the performance of dense SLAM (i.e., non-landmark based).
- **Focus:** pose-tracking algorithms in general, “scan matching” (ICP, NDT, ...) in particular.
- Many scan matching algorithms, but few characterizations:
 - ◆ Is the algorithm “optimal” in some sense?
 - ◆ If not, are there upper/lower bounds on its error?
- Approaches:
 1. Benchmarking (NIST, EURON sig, RAWSEEDS, etc.)
 2. Theoretical analysis:
 - ◆ Landmark-based, extended Kalman Filter setting.
(See works of Roumeliotis, Mourikis, Huang, and many others...)
 - ◆ What about algorithm-independent results, infinite-dimensional maps, dense sensors?

Summary

1. Cramér-Rao bound
2. Localization vs. pose tracking vs. mapping vs. SLAM
 - Localization is finite-dimensional and therefore “easy” to analyze
 - For pose tracking, one must deal with the unknown infinite-dimensional world (like SLAM)
3. Main results on pose tracking
 - Cramer-Rao bounds on the accuracy for pose tracking
 - Results independent of the map prior and representation
 - Valid for any relative sensor
4. Experiments with scan matching
 - ICP’s covariance is very close to the bound (almost optimal)
5. Ideas used in the proofs
 - intrinsic estimation, shape spaces, etc.

Cramér-Rao bound

Cramér-Rao bound. Suppose we are estimating the state \mathbf{x} based on the observations \mathbf{y} . Then, for any unbiased estimator $\hat{\mathbf{x}}$:

$$\text{cov}[\hat{\mathbf{x}}] \geq (\mathcal{I}[\mathbf{x}])^{-1} \quad (1)$$

where $\mathcal{I}[\mathbf{x}]$ is the Fisher Information Matrix (FIM):

$$\mathcal{I}[\mathbf{x}] = \mathbb{E} \left\{ \frac{\partial \log p(\mathbf{y}, \mathbf{x})}{\partial \mathbf{x}} \frac{\partial \log p(\mathbf{y}, \mathbf{x})}{\partial \mathbf{x}}^T \right\} \quad (2)$$

Notes:

- Elegant, no assumptions on the algorithm.
- It can also be stated for biased estimators, and in a Bayesian context.
- Strict for linear problems, and nonlinear problems at high SNR ratios.

Terminology

(Arbitrary) naming of problems:

Localization: estimate the pose in a known map.

Pose tracking: estimate the incremental displacement in a unknown map.

Mapping: given the poses, estimate the map.

SLAM: estimate both poses and map.

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↓ infinite dimensional ($\mathcal{I}[q, \text{map}]$ infinite dimensional)

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- This distinction is useful for the *analysis* of the problems, not the implementation.
- Everything is finite-dimensional if using landmarks.

Setup: one-shot pose tracking

Setup: Take observations y^a, y^b at the poses $q_*, q_* \oplus \delta$:

$$y^a = r(q_*, w) + \text{noise}$$

$$y^b = r(q_* \oplus \delta, w) + \text{noise}$$

Question: How precise can the estimate of δ be?

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$$y^b = r(q_* \oplus \delta, w) + \text{noise}$$

Question: How precise can the estimate of δ be?

The analysis is harder than for localization:

- The world w is unknown and part of the state.
- The world w is infinite dimensional ($\mathcal{I}[\delta, w]$ is an infinite matrix)
- How to get results independent of the representation for w ?
- How to correctly state the prior for w ?
 - ◆ representation \simeq prior: map composed of segments assigns zero prior to smooth curves, and vice versa.

Approach: look for the best bounds we can give, independently of w .

Main results

Cramer-Rao bound: $\text{cov}[\delta] \geq \mathcal{I}[\delta]^{-1}$, but $\mathcal{I}[\delta]$ depends on the world...

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Theorem: Without additional information on the prior for the world, the sensor model or the size of δ , the best bounds on $\mathcal{I}[\delta]$ are:

$$\mathbf{0} \leq \mathcal{I}[\delta] \leq \left(\mathcal{I}_{\text{loc}}(q_*)^{-1} + \mathcal{I}_{\text{loc}}(q_* \oplus \delta)^{-1} \right)^{-1}$$

where \mathcal{I}_{loc} is the information matrix for *localization* on a *known* map.

Corollary: this bound is strict for $\delta \rightarrow \mathbf{0}$.

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What makes this result interesting:

- The analysis for pose-tracking (infinite-dimensional) is reduced to localization (finite-dimensional).
- As $\delta \rightarrow \mathbf{0}$, the prior of the map does not matter (“the data is the model”).
- Valid for any relative sensor (range-finders, sonars, cameras), any algorithm.

Results for scan matching (ICP)

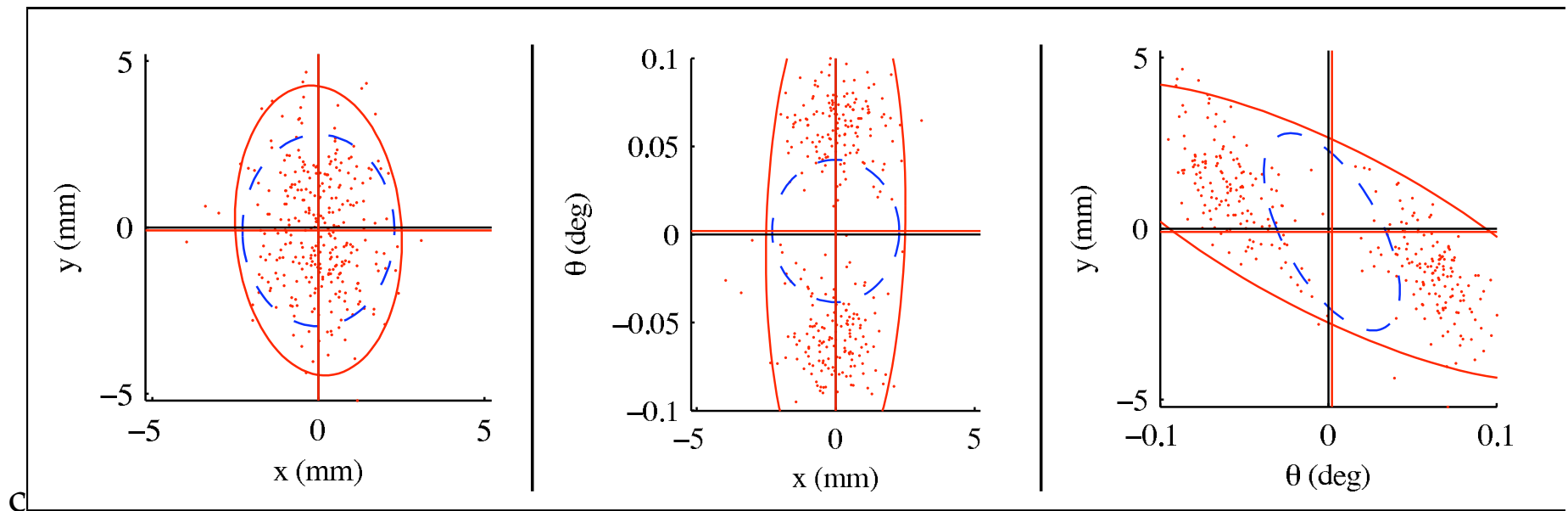
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2. This bound does not predict statistical bias.
3. The bound does not predict algorithmic errors (e.g., local minima).



Estimation error versus error bound on the (x, y) , (x, θ) , (y, θ) planes.

Ideas used in the paper

- Calculus of Fisher Information Matrices
 - ◆ (not to be confused with Information Filter, etc.)
- Intrinsic estimation (statistics compatible with differential geometry)
 - ◆ Any result must be independent of the parameterization chosen by a particular algorithm.
- Statistical geometry (statistics on shapes)
 - ◆ The of worlds can be factorized as

$$\text{World} = \text{Shape} \times \text{Pose}$$

- ◆ ex: one defines formally a “relative sensor” based on this factorization.

Conclusions

- We need stronger analytical results on dense localization/SLAM algorithms:
 - ◆ lower bounds on the error (depend on the problem)
 - ◆ upper bounds on the error (depend on the solution)

- The analysis for pose tracking has the same issues of SLAM:
 - ◆ The world is unknown and infinite-dimensional.
 - ◆ How to find results independent of the representation of the world?
 - ◆ How to express correctly the prior for the world?

- This paper found a way to sidestep these issues.
 - ◆ but these must be dealt with upfront for meaningful analysis of infinite-dimensional SLAM.

Analysis for dense localization

- Localization on known map is a *finite dimensional* problem, $\mathbf{x} = \mathbf{q} = (x, y, \theta)$.
- What is the achievable accuracy on estimating the pose?

The FIM for a single pose $\mathbf{q} = (x, y, \theta)$ is

$$\mathcal{I}[\mathbf{q}] = \sum_i^n \frac{1}{\sigma_i^2 \cos^2 \beta_i} \begin{bmatrix} \mathbf{v}(\alpha_i) \mathbf{v}(\alpha_i)^\top & r_i \sin \beta_i \mathbf{v}(\alpha_i) \\ * & r_i^2 \sin^2 \beta_i \end{bmatrix} + \text{prior}$$

- ◆ σ_i is the noise for the i -th reading.
- ◆ $\mathbf{v}(\alpha_i)$ is the surface normal at the point intercepted by the i -th reading.
- ◆ β_i is the incidence angle.

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- Do algorithms achieve this limit?
 - ◆ Not, e.g. ICP, or similar approaches, unless suitably modified.
 - ◆ Not the usual implementations of particle filters — they trade accuracy for robustness.

**Clint
Eastwood
is Dirty Harry in
Magnum
Force**



“A good man
knows his limitations”

— Harry Callahan