Real-valued average consensus over noisy quantized channels

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Consensus problems

- **Consensus:** reach the agreement of agent beliefs or agent states, respecting the given communication constraints.
- **Basic average consensus problem:** $x_i(k) \to \frac{1}{n} \sum x_i(0)$**.**

- Interesting to me because it is an example of **distributed computation** done by a network of simple units.
- Example success story of control-theory + computation:
  R. W. Brockett, "Dynamical Systems That Sort Lists, Diagonalize Matrices and Solve Linear Programming Problems," – magic formula: $\dot{H} = [H, [N, H]]$.
- Computation/control on distributed/noisy substrates will be an important topic:
  - neuronal networks (neuroscience)
  - noisy electronic components (precision vs. efficiency)
  - chemical reaction networks

Ideas from neuroscience

- The **brain** is the only instance of intelligence we know. We are very very far from understanding how it works.
- How about **neurons**?
  - Asynchronous distributed computation using **spikes.**
  - They are **slow** with respect to the dynamics they control (e.g. fruit fly).
  - They are **noisy.**
  - Lots of models (<u>we don't have a clue</u> of what is important)
    - Simplest non-trivial: linear sum of inputs + noisy nonlinearity.
- Can a control theorist tell something interesting?

<u>Useless</u> things to prove:

- "stability"
- "synchronization"

Interesting things to prove:

- computational properties
- adaptation/learning

- Can a noisy spiking network solve the consensus problem?

Some related work

Real-valued consensus over quantized channels is a two-part strategy:

1. *Communication strategy:* decide the value $y_j(k) \in \mathbb{Z}$ to send.
2. *Update strategy:* update the node's state $x_i(k)$ based on received $y_j(k)$

- [Aysal *et al* '07]: Given $\mathbf{P}$ stochastic, let

$$y_j(k) = q^p(x_j(k)) \qquad\qquad x_i(k) = \sum_j \mathbf{P}_{i,j}\, y_j(k)$$

Uses "probabilistic quantization" $q^p(x) = \begin{cases} \lceil x \rceil & \text{with probability } x - \lfloor x \rfloor \\ \lfloor x \rfloor & \text{otherwise} \end{cases}$

Results: consensus is reached to a value $\tau \in \mathbb{Z}$; $\mathbb{E}\{\tau\} =$ average.

- [Carli *et al.* '08]: Given $\mathbf{P}$ doubly stochastic, let

$$y_j(k) = \mathrm{round}(x_j(k)) \qquad\qquad x_i(k+1) = x_i(k) - y_i(k) + \sum_j \mathbf{P}_{i,j}\, y_j(k)$$

Results: the average is conserved; the consensus is not reached.

Model/approach

Update strategy: We adapt from [Olfati-Saber '07]:

$$x_i(k+1) = x_i(k) + \frac{\eta}{\Delta} \sum_j a_{ij} \left(y_j(k) - x_i(k)\right)$$

- $a_{ij} = a_{ji}$ is an element of the adjacency matrix; Δ is the degree of the graph; $\eta \in (0,1)$ a parameter.

Communication strategy

- Assume $y(k) = \boldsymbol{\psi}\left(x(k)\right)$, with $\boldsymbol{\psi}$ *arbitrary* function:

$$|\boldsymbol{\psi}(x) - x| \leq \beta$$

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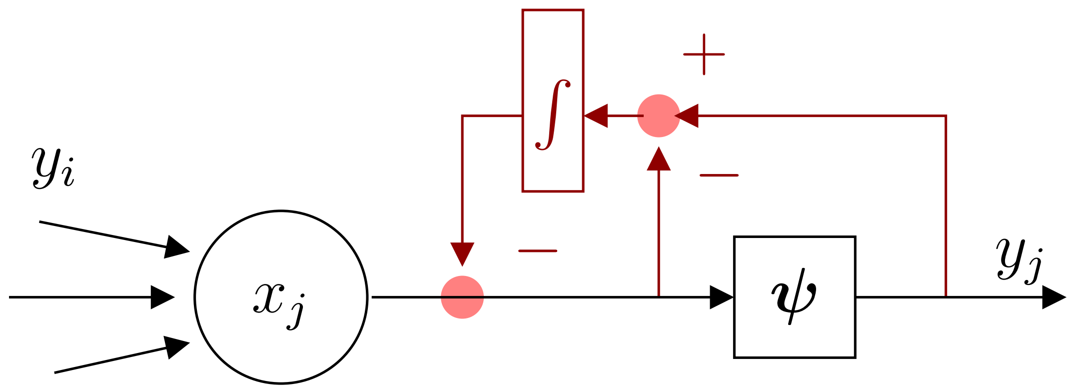

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Communication strategy

- Assume $y(k) = \boldsymbol{\psi}\left(x(k)\right)$, with $\boldsymbol{\psi}$ *arbitrary* function:

$$|\boldsymbol{\psi}(x) - x| \leq \beta$$

$$\begin{aligned} y_j(k) &= \boldsymbol{\psi}\left(x_j(k) - c_j(k)\right) \\ c_j(k+1) &= c_j(k) + \underbrace{\left(y_j(k) - x_j(k)\right)}_{\text{transmission error}} \end{aligned}$$

Has the flavor of a *self-inhibitory action potential.*

Behavior of the drift

- Define the drift $d(k)$ as

$$d(k) \triangleq \left| \frac{1}{n} \sum_i x_i(k) - \alpha \right|$$

  - $\alpha \triangleq \frac{1}{n} \sum_i x_i(0)$ is the goal state.

- **Proposition:** The drift is bounded:

$$d(k) \leq \eta\beta$$

  - β is the bound on the quantization error
  - η is the speed of the update strategy

- By choosing η, we can make the drift as small as desired.

Behavior of the disagreement error

- Take as an error measure the *average disagreement:*

$$\varphi(k) \triangleq \left[\frac{1}{n\Delta} \sum_{i,j} a_{ij} \left(x_i(k) - x_j(k) \right)^2 \right]^{1/2}$$

  - Δ is the degree of the graph ($n\Delta \simeq$ number of edges)

- **Proposition:** Eventually, the disagreement is bounded by:

$$|\varphi(k)| \leq \sqrt{6} \cdot \eta\beta \cdot \frac{\lambda_n\{\mathbf{L}\}}{\lambda_2\{\mathbf{L}\}}$$

  - $\lambda_2\{\mathbf{L}\}$ is the second smallest eigenvalue, $\neq 0$ if graph connected.
  - β is the bound on the quantization error
  - η is the speed of the update strategy

- By choosing η, we can make the disagreement as small as desired.

Comparison

Method	Drift	Disagreement
No quantization	$d(k) = 0$	$\varphi(k) \to 0$
Carli *et al.*	$d(k) = 0$	$\varphi(k) \to c > 0$
Aysal *et al.*	$d(k) \neq 0$	$\varphi(k) \to 0$
Proposed strategy	$d(k) \leq \eta\beta$	$\lim_{k\to\infty} \varphi(k) \leq c \cdot \eta\beta \frac{\lambda_n\{\mathbf{L}\}}{\lambda_2\{\mathbf{L}\}}$

- Therefore, consensus can be reached with arbitrary precision.
- But small η implies slow convergence.

Characterization of the bound

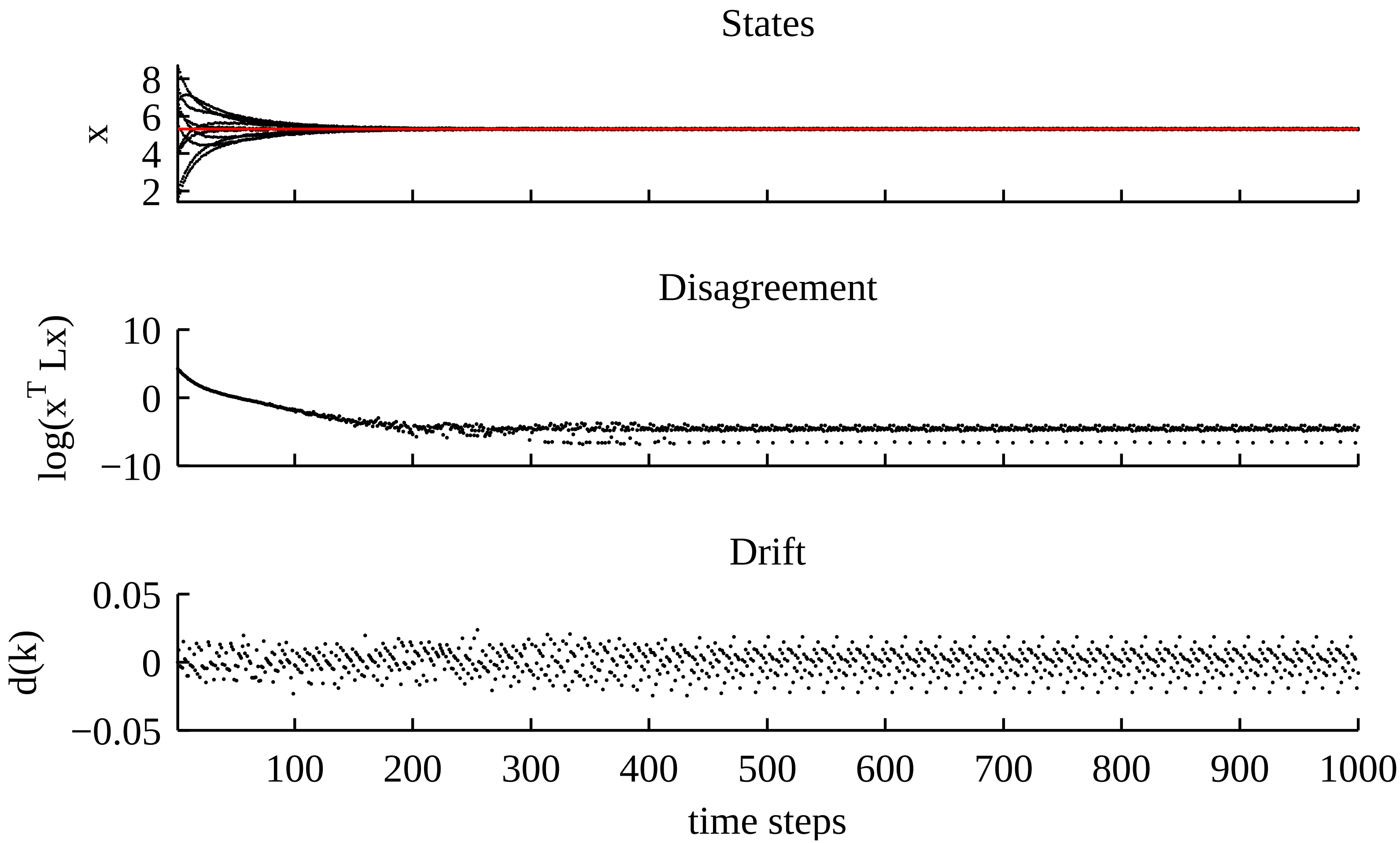
- For some graphs, $\lambda_n \mathbf{L} / \lambda_2 \mathbf{L}$ depends on the number of nodes n.
  - yet the performance appear to be largely independent of n

graph	$\lambda_n \mathbf{L}$	$\lambda_2 \mathbf{L}$	$\lambda_n \mathbf{L} / \lambda_2 \mathbf{L}$
star	n	1	n
complete	n	n	1
ring	4	$2 - 2\cos\left(\frac{2\pi}{n}\right)$	n^2
path	$2 + 2\cos\left(\frac{\pi}{n}\right)$	$2 - 2\cos\left(\frac{\pi}{n}\right)$	n^2

Examples

ψ = round; ring graph with $n = 10$ nodes.

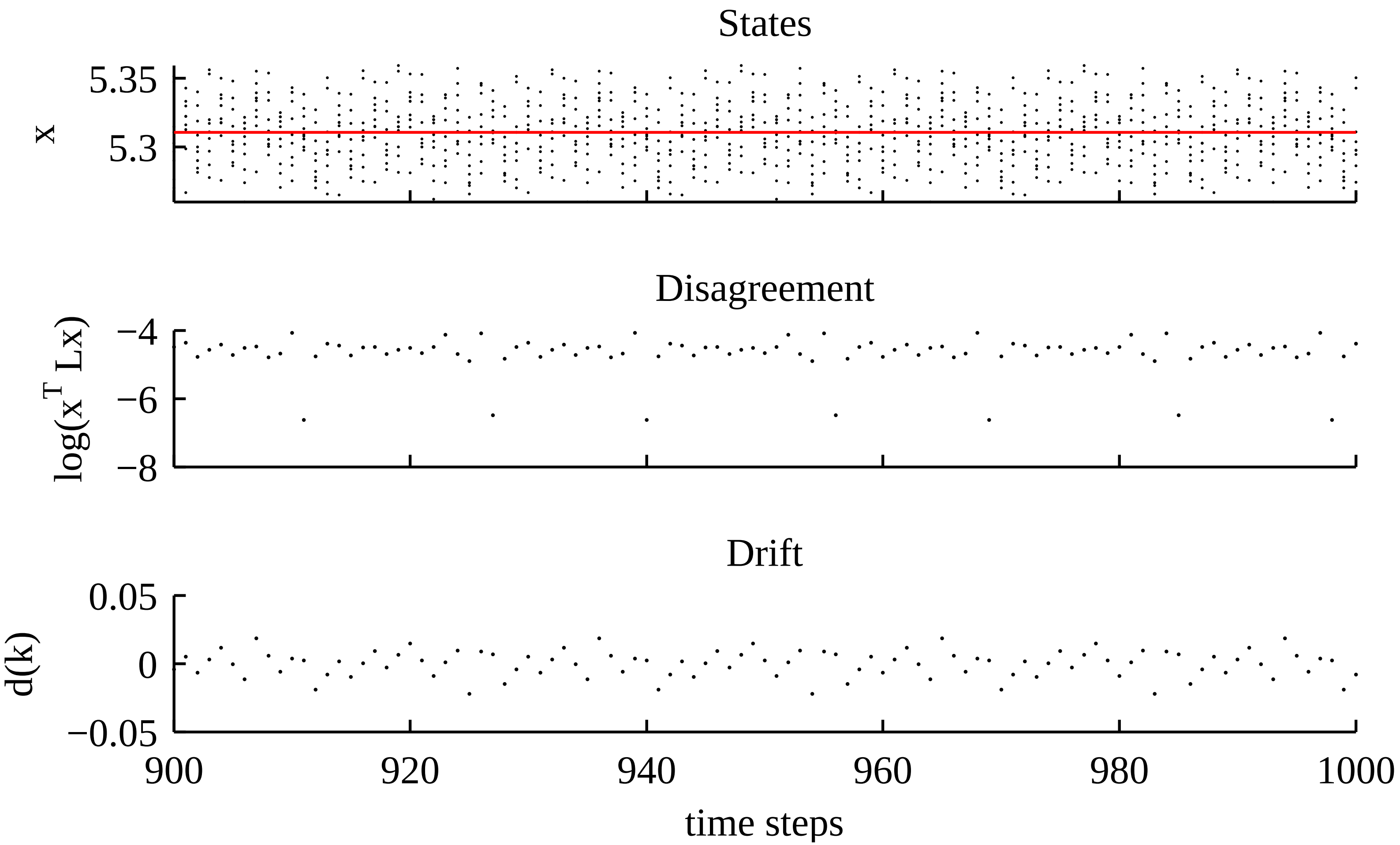
$\eta = 0.1$, overall behavior



Examples

ψ = round; ring graph with $n = 10$ nodes.

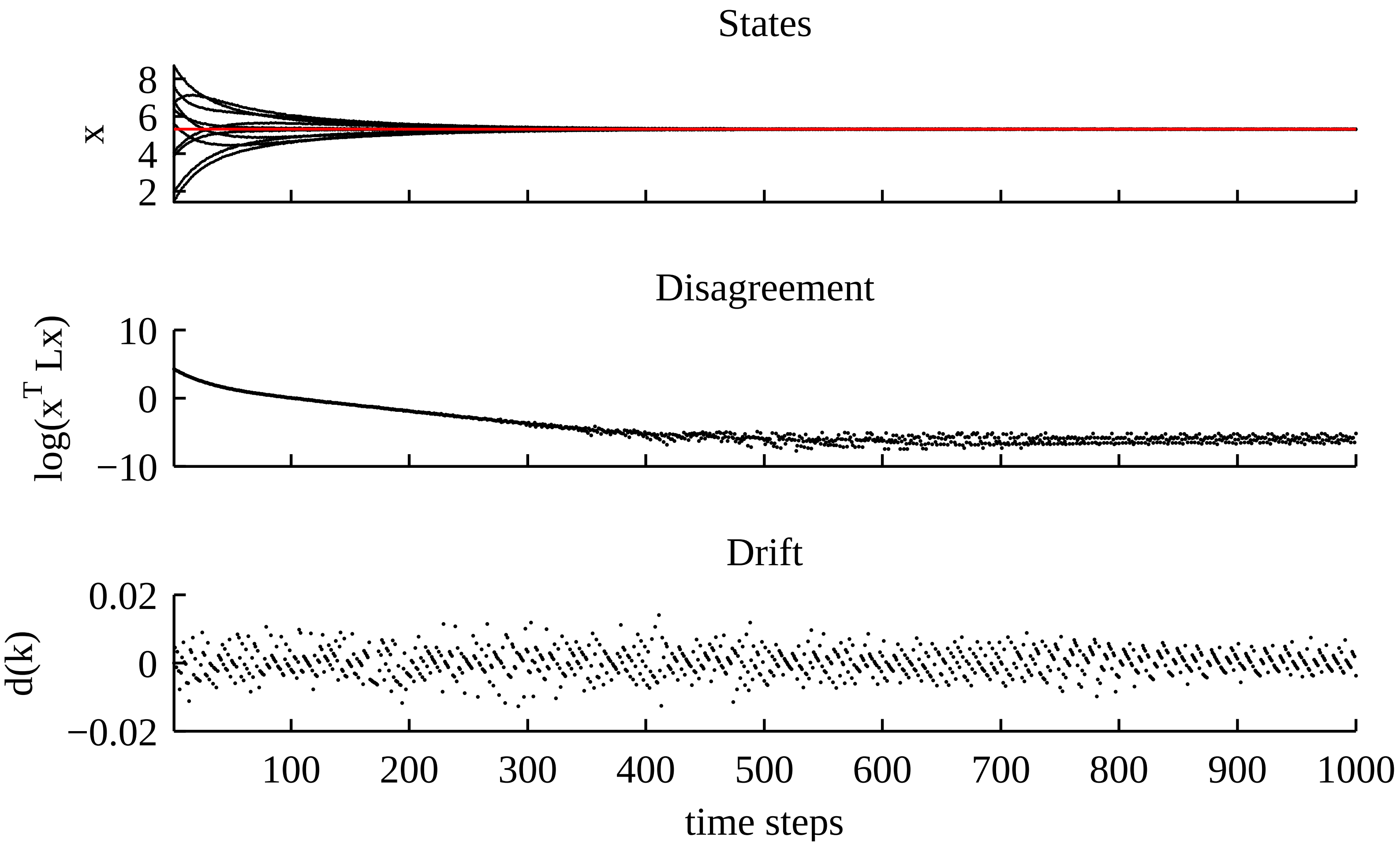
$\eta = 0.1$, last 100 steps



Examples

$\boldsymbol{\psi}$ = round; ring graph with $n = 10$ nodes.

$\eta = 0.05$, overall behavior



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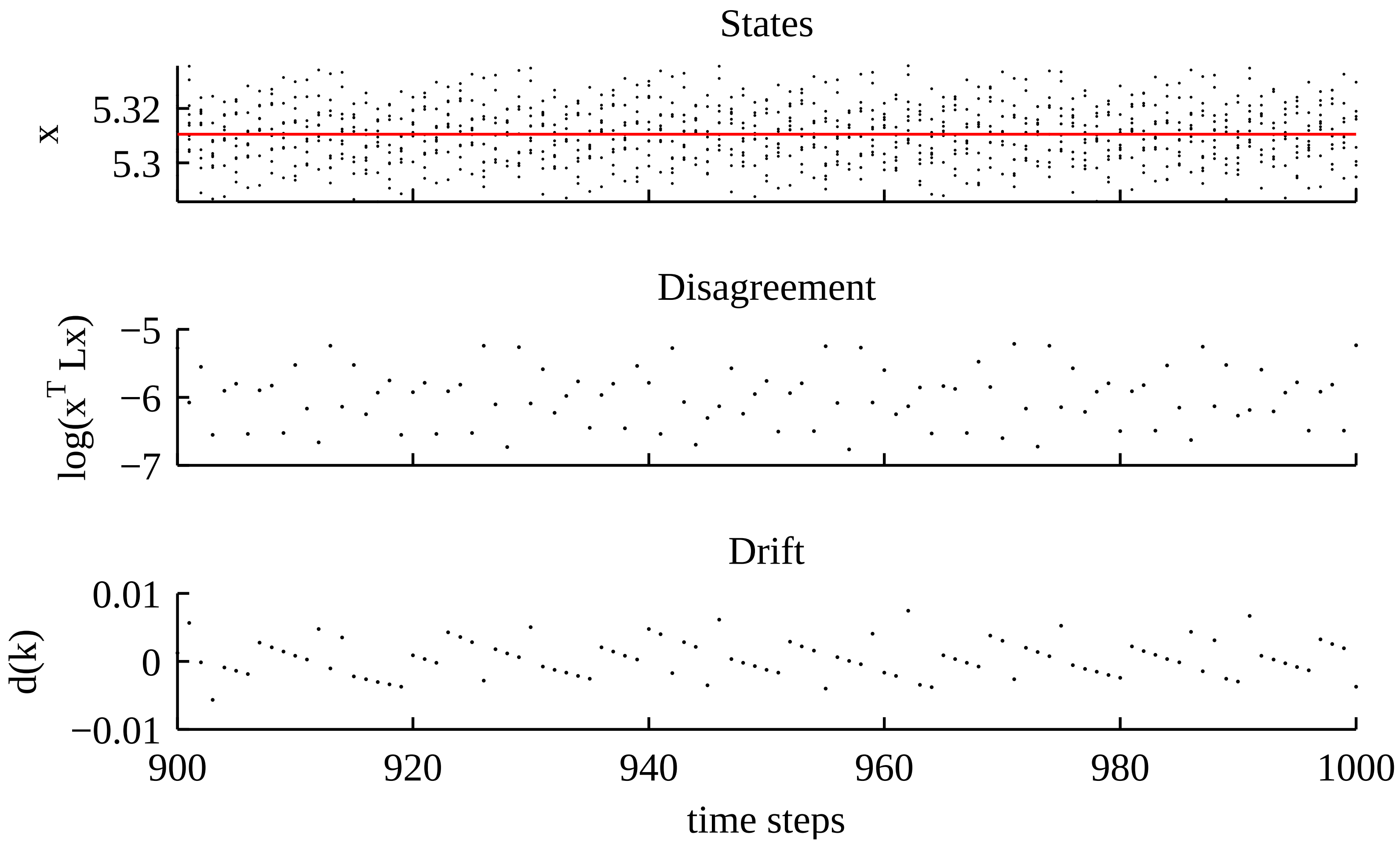

Conclusions

- Consensus can be reached with arbitrary precision regardless of quantization and noise.

- **Possible improvements:**
  - Characterization of convergence speed / precision tradeoffs with choosing η.
  - Find better bounds
    - In practice, the error appears independent of the number of nodes. However, $\lambda_n\{\mathbf{L}\}/\lambda_2\{\mathbf{L}\} \simeq O(n^2)$, for ring graphs.
    - Consider with specific quantization functions $\boldsymbol{\psi}$ or topologies.
  - Prove that, if $\boldsymbol{\psi}$ deterministic, it converges to a periodic orbit