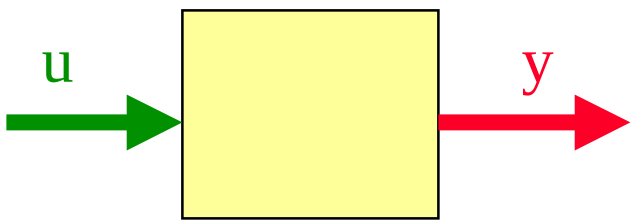
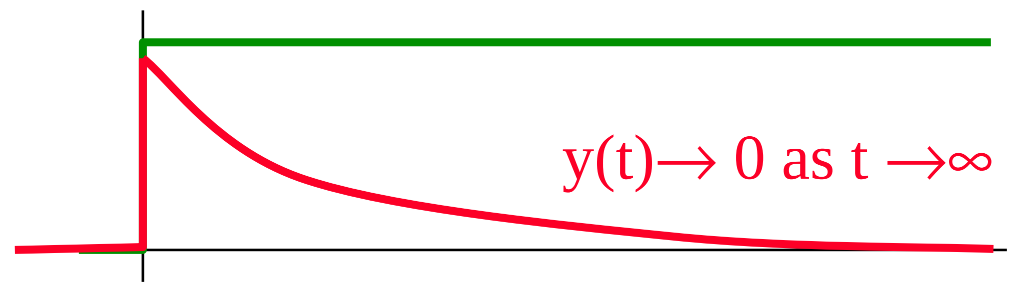
Integral feedback and asymptotic tracking

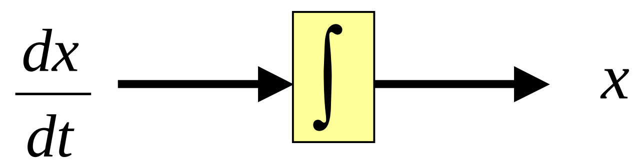
These are informal notes to help with intuition about integral feedback.



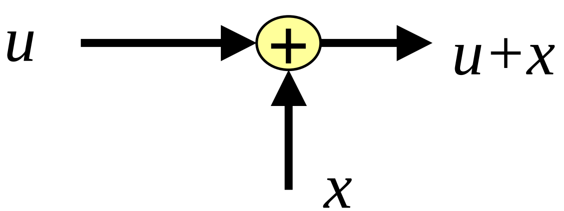


How can we robustly build this response out these components?

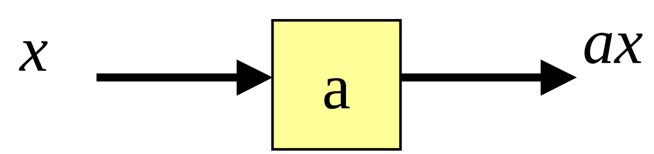
integrator

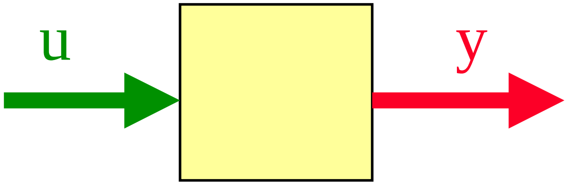


adder



constant gain

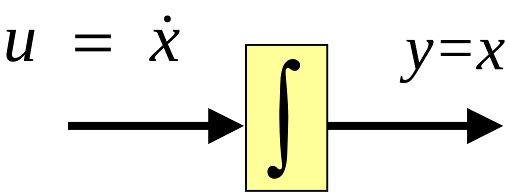




Components of linear systems

Notation: $\dot{x} \equiv \frac{dx}{dt}$

integrator

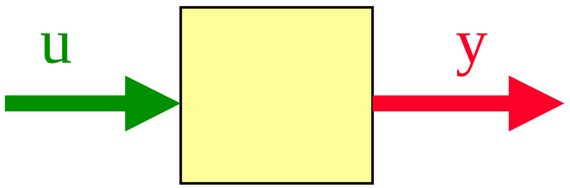


$$y(t) = y(0) + \int_0^t u(\tau)d\tau$$

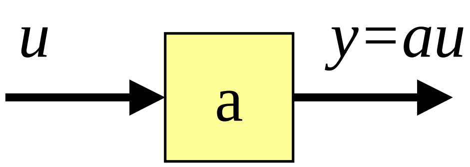
state-space:

$$\dot{x} = u$$
$$y = x$$

Components of linear systems

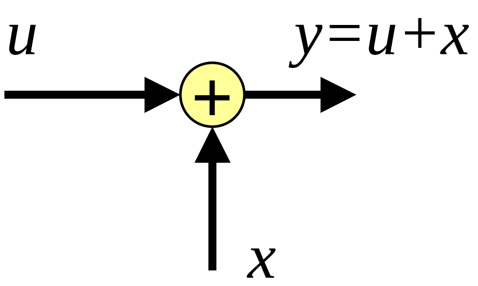


constant gain



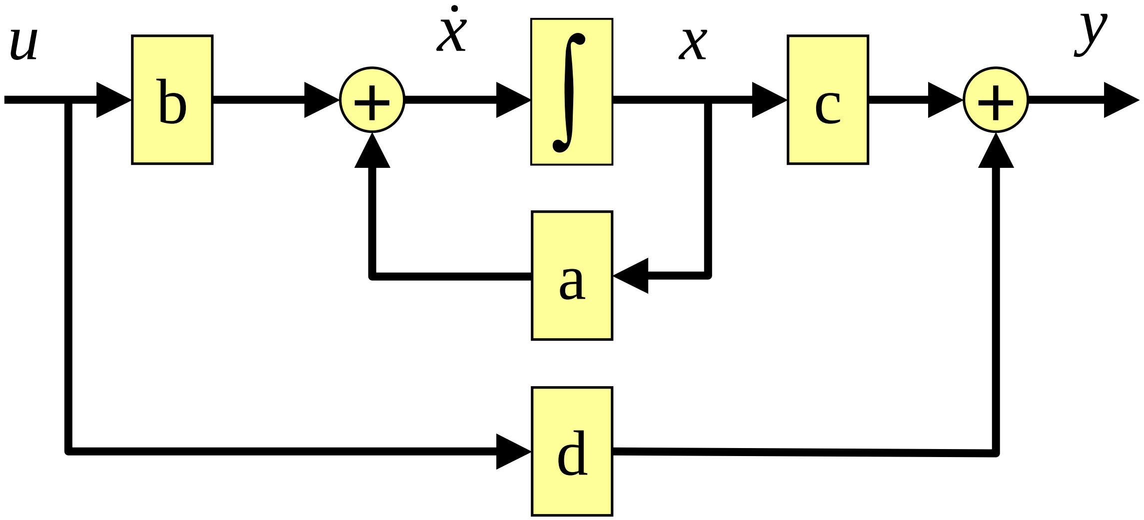
$$y = au$$

adder



$$y = u + x$$

First approach (general state space)



$$\dot{x} = ax + bu$$

$$y = cx + du$$

$y(t) \to 0$ as $t \to \infty$

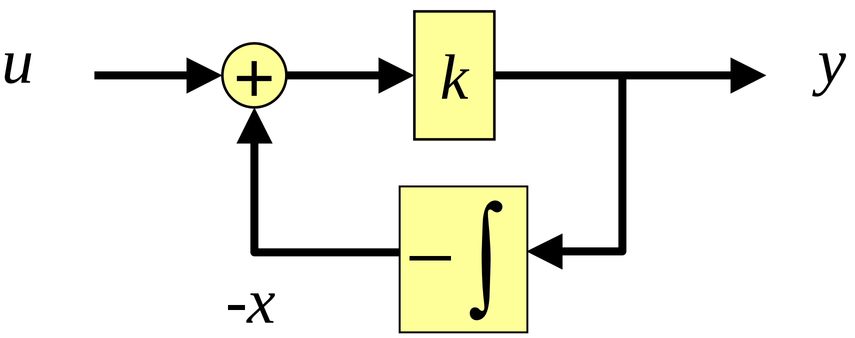
iff

- $a < 0$
- $ad - bc = 0$

requires "fine-tuning" of parameters

Second approach (integral feedback)

$$\dot{x} = y$$
$$y = k(u - x)$$



integral feedback

$y(t) \to 0$ as $t \to \infty$

iff

$k > 0$

requires no “fine-tuning”

Integral feedback, cont.

Similarly, if

$$\dot{x} = y$$
$$y = cx + du$$

u constant

$y(t) \to 0$ as $t \to \infty$

iff

$c < 0$

Need stability of $\dot{x} = y = cx + du$

Comparison

$$\dot{x} = ax + bu$$
$$y = cx + du$$

$y(t) \to 0$ as $t \to \infty$
iff
• $a < 0$
•ad-bc=0

requires
parameter
tuning to
get ad-bc=0

$$\dot{x} = y$$
$$y = cx + du$$

$y(t) \to 0$ as $t \to \infty$
iff
$c < 0$

requires
specific
structure

often easier to build

Comparison

$$\dot{x} = y$$
$$y = cx + du$$

What is “structure”?
Here, it is simply a pattern of zero and nonzero parameters, with the assumption that only zero values can be robustly constructed. In other words, if all nonzero parameters are uncertain, *structure* is then the specification of which parameters are zero.

$y(t) \to 0$ as $t \to \infty$
iff
$c < 0$

requires specific *structure*

often easier to build

But…

$$\dot{x} = ax + bu$$
$$y = cx + du$$

$$\dot{x} = y$$
$$y = cx + du$$

The theorem says that if this does hold, then there is a hidden integral feedback structure.

y(t)$\rightarrow$ 0 as t $\rightarrow\infty$
iff
• a < 0
•ad-bc=0

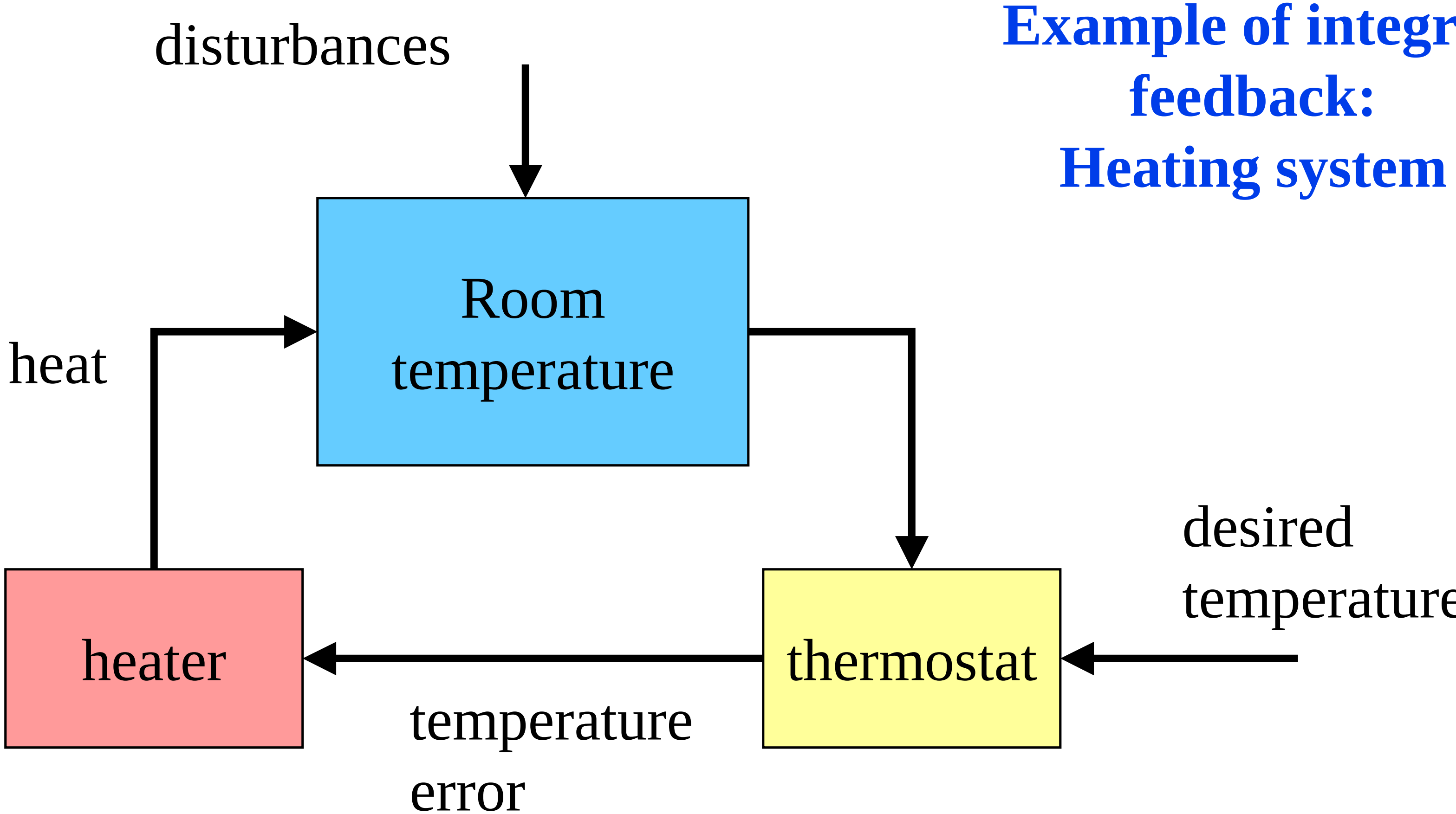
requires parameter tuning to get ad-bc=0

y(t)$\rightarrow$ 0 as *t* $\rightarrow\infty$
iff
c < 0

requires specific *structure*

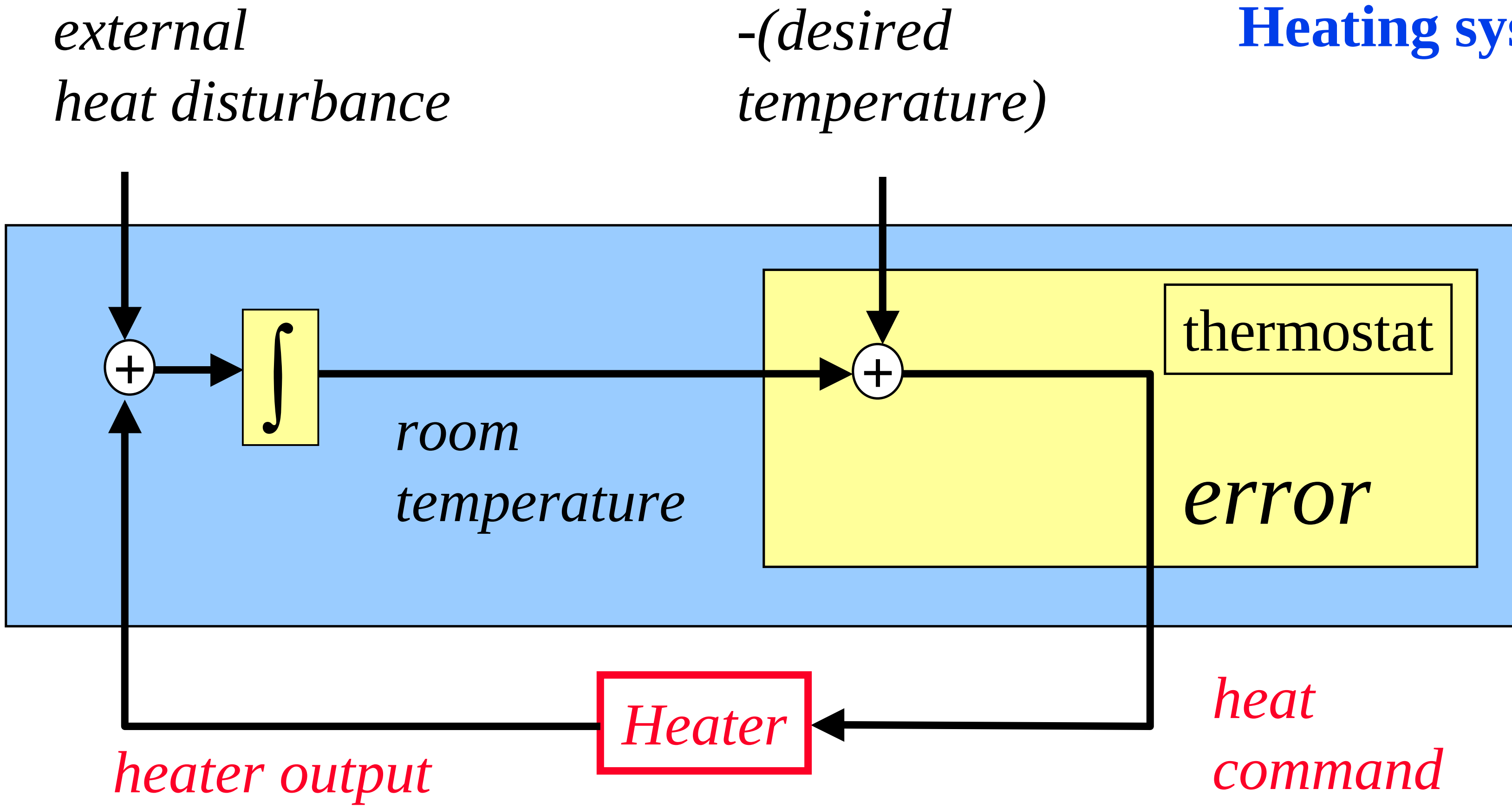
often easier to build

Example of integral feedback: Heating system



$$temperature \propto \int heat \quad \Rightarrow \quad error \xrightarrow{t \to \infty} 0$$

Example of integral feedback: Heating system

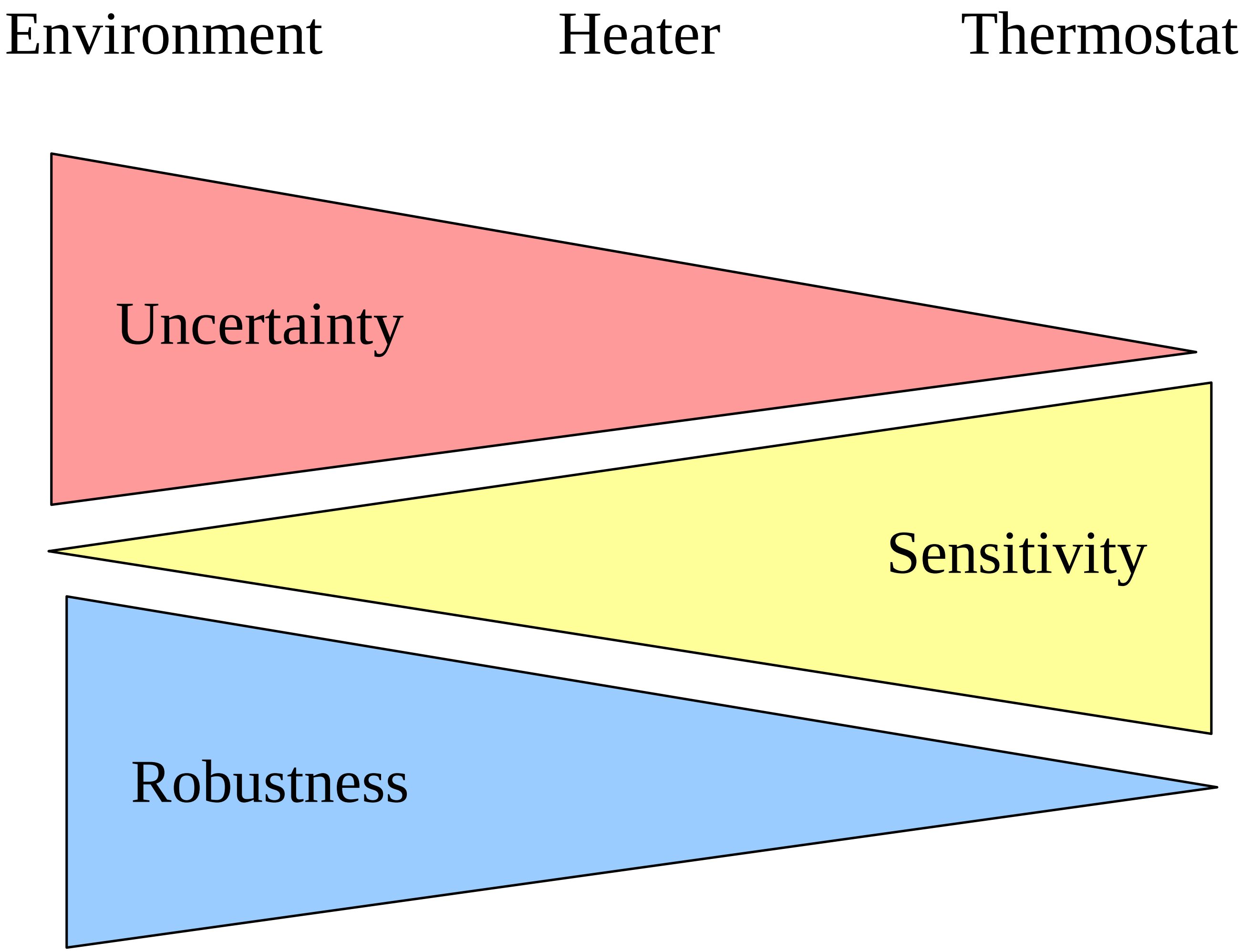


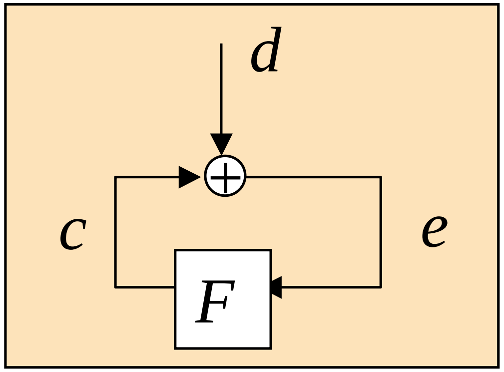
$$\text{temperature} \propto \int \text{heat} \quad \Rightarrow \quad \text{error} \xrightarrow{t \to \infty} 0$$

Uncertainty and robustness in heating system

- The heating system has a wide range of uncertainty in its components, and a wide range of robustness.
- **Environment**: The environment is very uncertain, with large ranges of external temperatures and variations due to people, doors and windows open, etc. The system is quite robust to these variations.
- **Heater**: The heat output of a heater varies within a range, and the system will robustly track temperature if this range isn't too great.
- **Thermostat**: It is relatively easy to build reliable and accurate thermostats. The overall system is very sensitive to any errors or malfunctions in the thermostat.

Uncertainty and robustness in heating system





Abstraction of heating system

d = disturbances and setpoints
e = error
c = compensation, control

$$c = F(e) = -k\int e$$

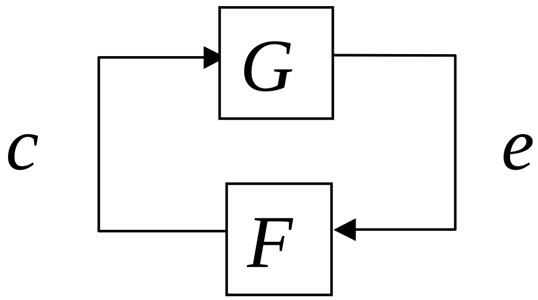
integral feedback

$$\Rightarrow \quad \dot{c} = -ke = -k(d + c)$$

steady state: $\dot{c} = 0$

$e(t) \to 0$ as $t \to \infty$
iff
$k > 0$

General integral feedback



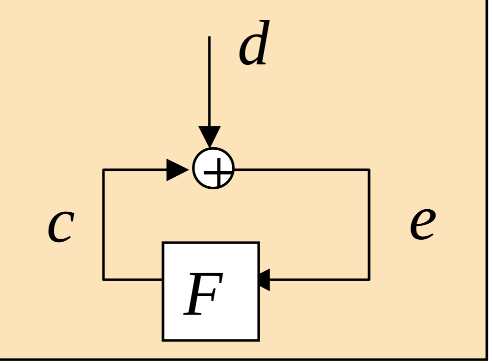
G = arbitrary system
e = error
c = "control"

$$F(e) = -k \int e$$

$$\dot{c} = -ke$$
$$= -k(d + c)$$

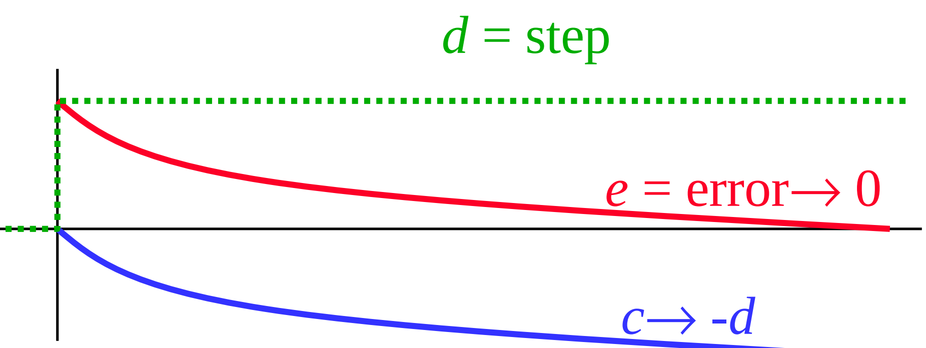
steady state: $\dot{c} = 0$

$e(t) \rightarrow 0$ as $t \rightarrow \infty$
if
complete system reaches equilibruim



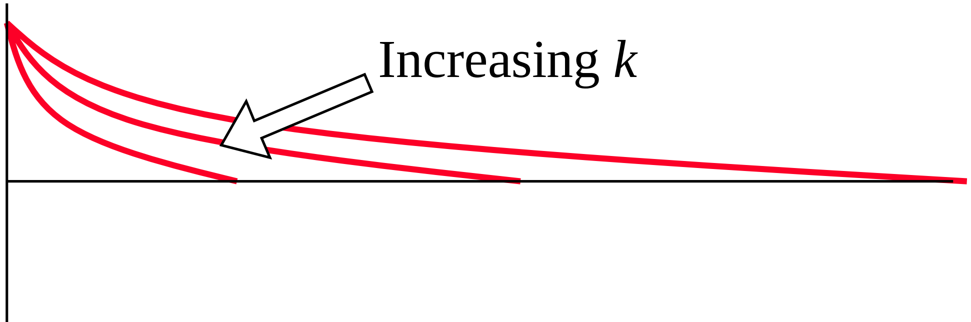
General properties of integral feedback

$$F(e) = -k\int e \quad \Rightarrow \quad \dot{c} = -k(d + c)$$



The error to a step asymptotically approaches 0, for any $k > 0$.

This is perhaps the most basic "result" in control theory and would appear as early as the sophomore level.



The gain k effects speed but not asymptotic error.

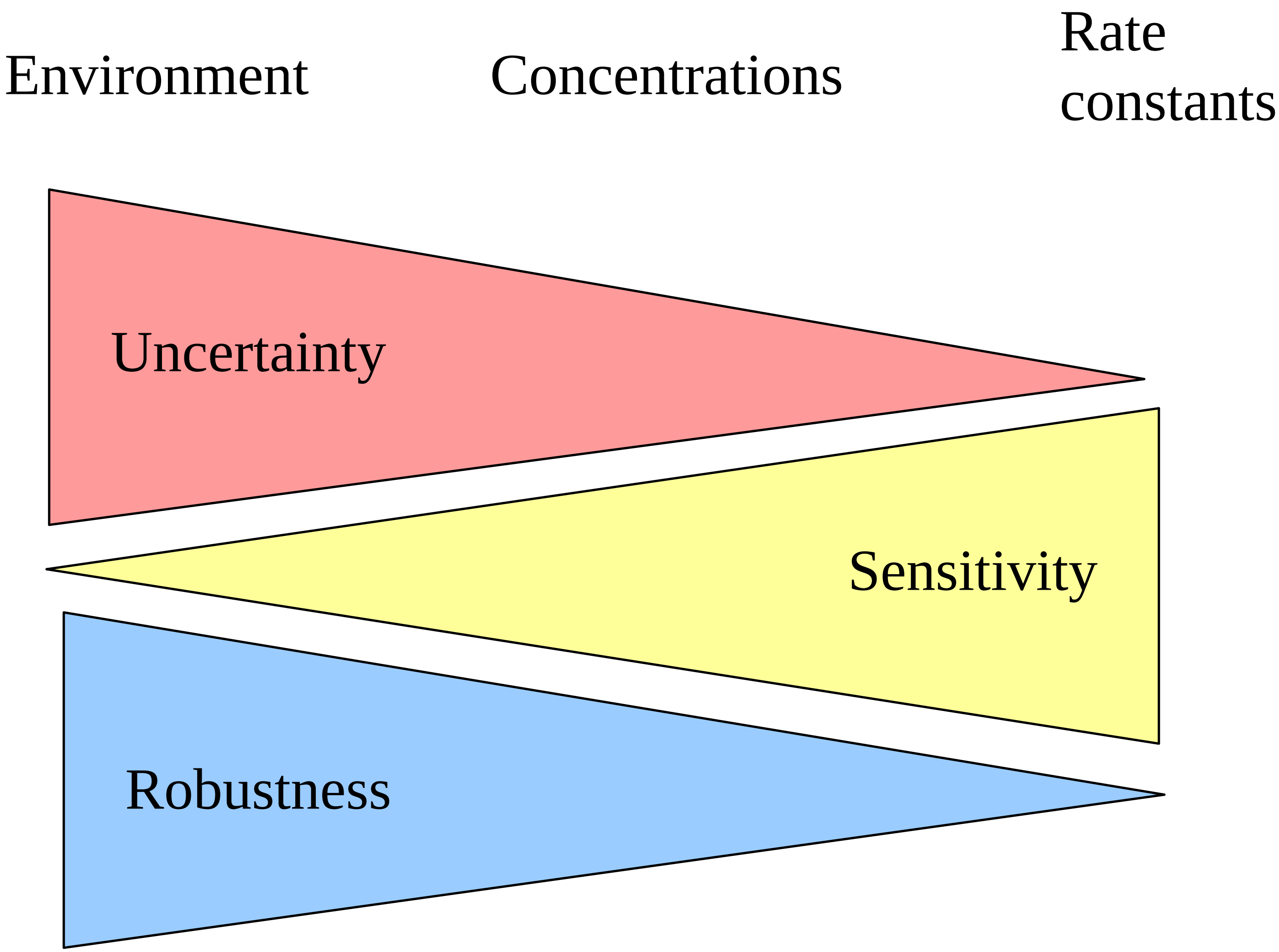
Integral feedback

- Most elementary result in control theory.
- Used ubiquitously in engineering systems: control of temperature in building, speed in cars, altitude, speed, and heading in airplanes, etc etc
- Number of integral feedback loops in some systems is large:

  >2000 in a typical high rise HVAC system

  >10,000 in typical refinery
- US power grid has >3000 generators supplying >100M customers, maintains voltage, frequency, and integral of frequency error (uses generalization of integral feedback).
- “Adaptive control” is also used, but it is different than integral control, and is beyond the scope of this discussion.

Uncertainty and robustness in chemotaxis



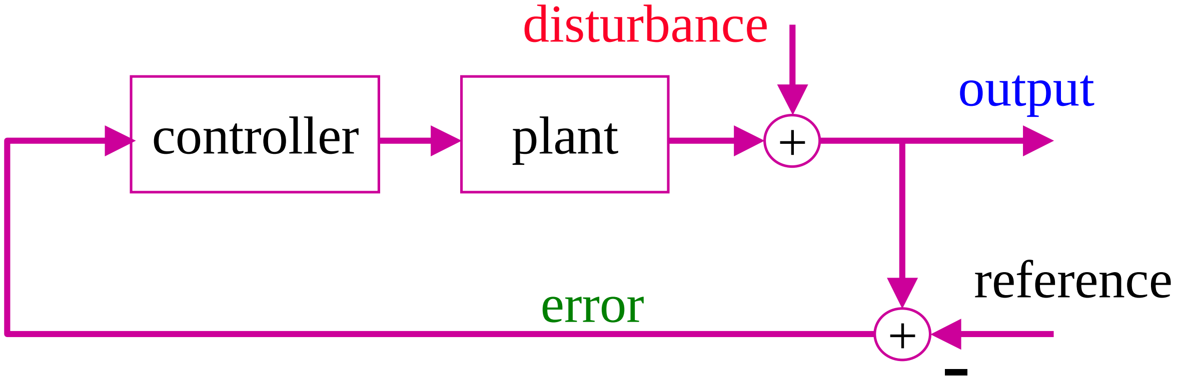
Related issues in asymptotic tracking, integral feedback, etc...

Special case of the **internal model principle**, which is, informally,

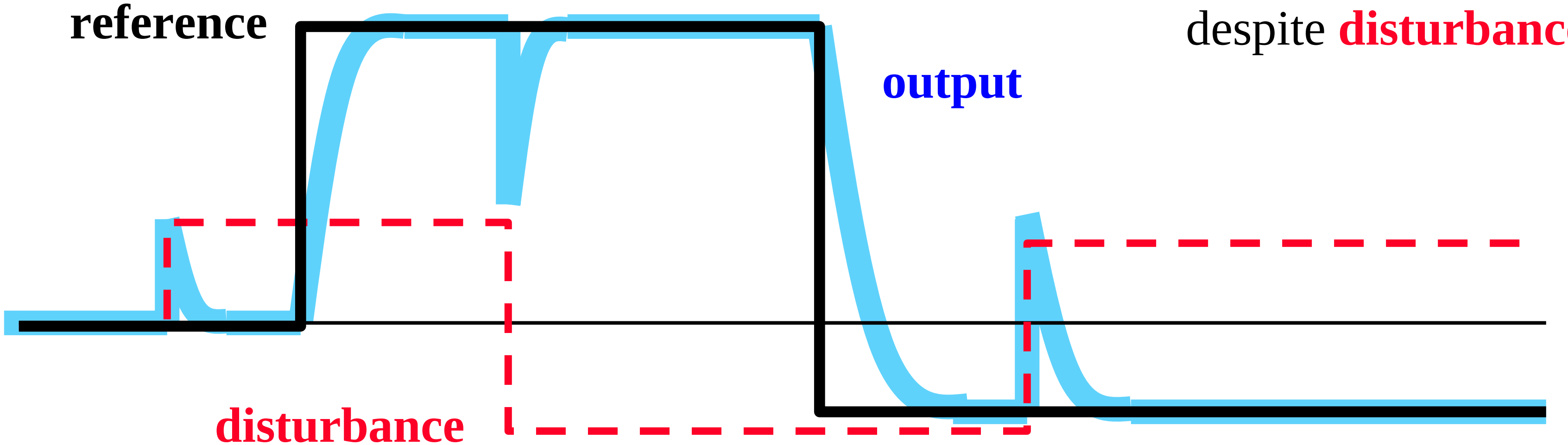
for asymptotic tracking of a signal,
the controller must contain a model of that signal

Related topics taught in introductory controls courses:

- effects of rate and magnitude saturation
- effects of measurement noise
- further constraints on achievable performance due to fixed parts of “plant”
- adaptation (which is something in addition to all this…)
- etc. etc.



Why do we call this "asymptotic tracking?"



The **output** ***asymptotically tracks*** the **reference** despite **disturbances**.

"Adaptive" usually refers to something more sophisticated, although the term is often ambiguous.