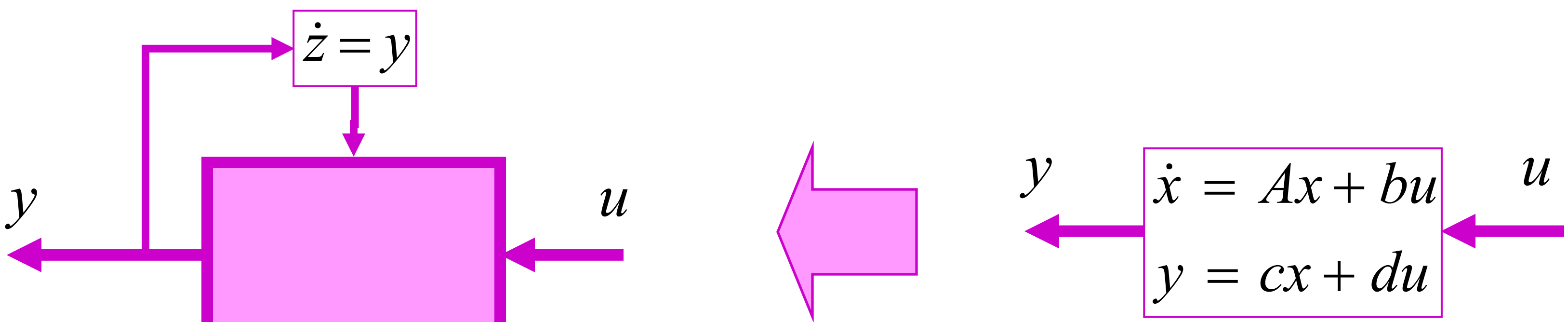
Integral feedback theorem

$$\dot{x} = Ax + bu$$
$$y = cx + du$$

Assume:

1) A stable
2) $b \neq 0$
3) $\begin{bmatrix} c & d \end{bmatrix} \neq 0$

$$\left.\begin{array}{c} y = 0 \\ \text{for all} \\ \text{constant } u \end{array}\right\} \Leftrightarrow \left\{\begin{array}{c} \exists\ z = kx \\ \ni \\ \dot{z} = y \end{array}\right.$$



This is the simplest nontrivial theorem in controls. It is called integral feedback because z is the integral of y, and the rest of the system must depend on z, since A is stable. Thus, integral feedback.

The proof is completely elementary, and is on the next page.

$$\dot{x} = Ax + bu$$
$$y = cx + du$$

Steady-state $\Rightarrow$

$$0 = Ax + bu$$
$$y = cx + du$$

so y=0 for all constant u, if and only if

$$\forall u, \exists x \ni \begin{bmatrix} A & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0$$

Assume A stable, and $b \neq 0$ $\Rightarrow$

$$\forall u \quad \left. \begin{aligned} x &= -A^{-1}bu \\ y &= (d - cA^{-1}b)u = 0 \end{aligned} \right\} \Leftrightarrow (d - cA^{-1}b) = 0$$

Assume $[\mathrm{c} \quad \mathrm{d}] \neq 0$ $\Rightarrow$

$$(d - cA^{-1}b) = 0 \quad \Leftrightarrow \quad \det \begin{bmatrix} A & b \\ c & d \end{bmatrix} = 0$$

$$\Leftrightarrow \exists k \neq 0 \quad \ni \quad [k \quad -1] \begin{bmatrix} A & b \\ c & d \end{bmatrix} = 0$$

$$\Leftrightarrow \exists k \neq 0 \quad \ni \quad k[A \quad b] = [c \quad d]$$

$$\therefore \quad \left. \begin{array}{c} y = 0 \\ \text{for all} \\ \text{constant } u \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} \exists\, z = kx \\ \ni \\ \dot{z} = y \end{array} \right.$$

This proof immediately extends to the case of uncertain systems, suppose we have

$$M \triangleq \begin{bmatrix} A & b \\ c & d \end{bmatrix} \quad M \in \mathfrak{M}$$

where $\mathfrak{M}$ is some set

$$\exists k : R^{(n+1)\times(n+1)} \to R^{(n+1)} \ni \forall M \in \mathfrak{M}$$
$$k(M) \in R^{n}, k(M) \neq 0$$

$$k(M)\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} c & d \end{bmatrix} \quad \forall M \in \mathfrak{M}$$

Let $z = k(M)y$, then

$$\dot{z} = k(M)\dot{x} = k(M)(Ax + bu)$$
$$= cx + du = y$$

(This holds for all $M \in \mathfrak{M}$)