CDS Technical Report 01-009
http://www.cds.caltech.edu/~murray/papers/dmp01-cds.html

Primitives for Human Motion: a Dynamical Approach

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September 22, 2001

Abstract

Using tools from dynamical systems theory and systems identification theory we develop the study of primitives for human motion which we refer to as *movemes*. We introduce basic definitions of *dynamical independence* of LTI systems and *segmentability* of signals and we develop classification and segmentation algorithms for two dimensional motions. We test our ideas on data sampled from four human subjects who were engaged in a simple real-life activity including two movemes. Our experiments show that we are able to distinguish between the two movemes and recognize them even when they take place in an activity containing more than one moveme.

1 Introduction

Building systems that can detect and recognize human actions and activities is an important goal of modern engineering. Applications range from human-machine interfaces, to security to entertainment. The first fundamental problem in achieving this goal is one of representation. Our point of view is that human activity should be decomposed into its building blocks which belong to an "alphabet" of elementary actions that the machine knows. We refer to these primitives of motion as *movemes*. This word first came up in the work by (Bregler and Malik, 1997). Their approach does not include an input and therefore is only applicable to periodic or stereotypical motions, such as walking or running where the motion is always the same. (Goncalves *et al.*, 1998) also proposed to divide human motion into elementary trajectories called movemes. They dealt with the problem in a phenomenological and non-causal way: each moveme was parameterized by goal and style parameters. We attempt here to define movemes in terms of causal dynamical systems; this way a moveme could be parameterized by a small set of dynamical parameters and by an input which drives the overall dynamics. Our aim is to build an "alphabet of movemes" which one can compose to represent and describe human motion similar to the way phonemes are used in speech. Two more problems we address are the ones of segmentation and classification: can a continuous trajectory of the human body be decomposed automatically into its component movemes?

We validate our ideas by analyzing the mouse trajectories generated by computer users as they "point-and-click" (we call this the reach moveme) and trace straight lines (we call this the draw moveme).

2 Axiomatic Perspective on Movemes

This section is concerned with the theoretical aproach to the study of movemes: we give a few basic definitions and set up the requisite mathematical framework. Let $M = M(\theta)$ denote a class of linear time-invariant (LTI) dynamical systems parameterized by $\theta \in \mathbb{R}^m$ and let $\mathcal{U}$ denote the class of allowable inputs. Let $y(t) = O(M(\theta)|_{u,Y_0})$ denote the output of $M(\theta)$ once parameter $\theta \in \mathbb{R}^p$, input $u \in \mathcal{U}$ and initial condition $Y_0 \in \mathbb{R}^n$ have been defined.

Definition 2.1 Let $M^R(\theta) = M(\theta),\ \theta \in C^R$ and $M^D(\theta) = M(\theta),\ \theta \in C^D$ denote two subsets of models in the class M parameterized respectively by $\theta \in C^R$ and $\theta \in C^D$ with $C^j \subset \mathbb{R}^m$, for $j = R, D$. The two sets $M^R(\theta)$ and $M^D(\theta)$ are said to be *dynamically independent* if:

(i) the class M and the class of inputs $\mathcal{U}$ are such that $O(M(\theta_1)|_{u_1,Y_0}) = O(M(\theta_2)|_{u_2,Y_0})$ iff $(\theta_1, u_1) = (\theta_2, u_2)$ for $u_1 \in \mathcal{U}$ and $u_2 \in \mathcal{U}$;

(ii) the sets C^R and C^D are non-empty, bounded and linearly separable.

A set $\mathcal{M} = \{M^1, ..., M^l\}$, where $M^i(\theta) = M(\theta),\ \theta \in C^i$ and $M^i \subset M$, is said to be a set of *mutually dynamically independent* model sets if all the pairs $\{M^i, M^j\}$ are dynamically independent for $i, j \in [1, l]$.

The linear separability requirement for the sets C^R and C^D can be relaxed, in a more general framework, just to separability.
Each of the elements of the set $\mathcal{M}$ of mutually dynamically independent model sets is called a *moveme*. Let $M^j \in \mathcal{M}$ be a moveme. We let $y^j(t) = O(M^j(\theta)|_{u,Y_0}) = O(M(\theta)|_{u,Y_0})$ denote the *moveme output* for M^j once the parameters $\theta \in C^j$, input $u \in \mathcal{U}$ and initial conditions Y_0 are determined. Given a signal $y(t)$, $t \in [t_0, N]$, let $s_1(t)$ and $s_2(t)$ be the two signals defined as

$$\begin{aligned} s_1(t) = y(t),\ t \in [t_0, n] \\ s_2(t) = y(t),\ t \in [n, N] \ , \end{aligned} \tag{1}$$

where $n \in (t_0, N]$. We let $(s_1(t), s_2(t))_n$ denote the *segmentation* of $y(t)$ at time n.

Definition 2.2 A signal $y(t)$ is said to be *segmentable* if there exists $n^* \in (t_0, N)$ such that the segmentation at time n^*, $(s_1(t), s_2(t))_{n^*}$, satisfies

$$\begin{aligned} s_1(t) &= O(M(\theta_1)|_{u_1,Y_{t_0}}),\ t \in [t_0, n^*] \\ s_2(t) &= O(M(\theta_2)|_{u_2,Y_{n^*}}),\ t \in [n^*, N] \end{aligned} \tag{2}$$

for some u_1, u_2, , Y_{t_0}, Y_{n^*}, θ_1, θ_2 with $(\theta_1, u_1) \neq (\theta_2, u_2)$. The couple $(s_1(t), s_2(t))_{n^*}$ defined here is refered to as the *actual segmentation*.

Proposition 2.1 A moveme output $y^i(t) = O(M^i(\theta)|_{u,Y_0})$, $t \in [t_0, N]$, is not segmentable.
Proof. Let $(s_1(t), s_2(t))_n$ be the segmentation of $y^i(t)$ for any $n \in (t_0, N)$. Suppose for $t \in [t_0, n]$ $s_1(t) = O(M(\theta_1)|_{u_1,Y_{t_0}})$ and $s_2(t) = O(M(\theta_2)|_{u_2,Y_n})$, $t \in [n, N]$. Also $s_1(t) = O(M^i(\theta)|_{u,Y_{t_0}})$, $t \leq n$ and $s_2(t) = O(M^i(\theta)|_{u,Y_n})$,$t \geq n$. Therefore $O(M(\theta_1)|_{u_1,Y_0})$= $O(M(\theta)|_{u,Y_0})$, $O(M(\theta_2)|_{u_2,Y_n}) = O(M(\theta)|_{u,Y_n})$ which by (i) of Definition 2.1 implies $(\theta_1, u_1) = (\theta, u)$, $(\theta_2, u_2) = (\theta, u)$ which in turn imply $(\theta_1, u_1) = (\theta_2, u_2)$ that contradicts Definition 2.2. □

Proposition 2.2 If $y(t)$, $t \in [t_0, N]$, is segmentable, then the actual segmentation is unique.
Proof. Let $(s_1(t), s_2(t))_{n^*}$ as defined in (2) be the actual segmentation of $y(t)$. Suppose there is an $n < n^*$ such that $(\tilde{s}_1(t), \tilde{s}_2(t))_n$ is an actual segmentation, then since $n < n^*$ we have $\tilde{s}_1(t) = O(M(\theta_1)|_{u_1,Y_{t_0}})$, $t \in [t_0, n]$ and $\tilde{s}_2^a(t) = O(M(\theta_1)|_{u_1,Y_n})$ for $t \in [n, n^*]$ while $\tilde{s}_2^b(t) = O(M(\theta_2)|_{u_2,Y_{n^*}})$ for $t \in [n^*, N]$ which means by Definition 2.2 that $\tilde{s}_2(t)$ is segmentable. Therefore $(\tilde{s}_1(t), \tilde{s}_2(t))_n$ is not an actual segmentation according to Definition 2.2. The same argument holds for $n > n^*$. □

In this work, the choice of the model class M is restricted to second order linear systems described by

$$\ddot{y}(t) = \theta^T \varphi(t) \ , \tag{3}$$

where $\theta \in \mathbb{R}^3$ and $\varphi^T(t) = (-\dot{y}(t), -y(t), u(t))$, with input $u(t) = 1(t) \in \mathbb{R}$. Given any signal $y(t)$, $t \in [t_0, N]$ whose dynamics is driven or not by (3), we can determine the best representative of $y(t)$ in the class M by minimizing $\int_{t_0}^{N} (\ddot{y}(t) - \theta^T \varphi(t))^2 dt$ with respect to θ, so that

$$\hat{\theta} = \text{argmin} \int_{t_0}^{N} (\ddot{y}(t) - \theta^T \varphi(t))^2 dt \tag{4}$$

to get

$$\ddot{\hat{y}}(t) = \hat{\theta}^T \hat{\varphi}(t) \quad \hat{\varphi}(t_0) = \varphi(t_0) \ , \tag{5}$$

where $\hat{\varphi}(t) = (-\dot{\hat{y}}(t), -\hat{y}(t), 1(t))^T$. We verify that the class (3) satisfies property (i) of Definition 2.1 by using the following lemmas.

Lemma 2.1 For any C^2 time signal $y(t)$ given by

$$\begin{cases} \dot{x} = \begin{pmatrix} 0 & 1 \\ -\theta_2 & -\theta_1 \end{pmatrix} x + \begin{pmatrix} 0 \\ \theta_3 \end{pmatrix} + \begin{pmatrix} 0 \\ d(t) \end{pmatrix} \\ y = (1, 0)x \end{cases} \tag{6}$$

with $d(t)$ white noise or $d(t) = 0$, let λ_1,λ_2 denote the eigenvalues for system (6), $E = \{(v_1, v_2)^T \in \mathbb{R}^2 : v_2 = \lambda(v_1 - \frac{\theta_3}{\theta_2}),\ \lambda = \lambda_1,\ \lambda_2\}$ and $\varphi(t)^T = (-\dot{y}(t), -y(t), 1(t))$. If $x(t_0) \notin E$, then the matrix $\int_{t_0}^{t_1} \varphi(t)\varphi(t)^T dt$ is nonsingular.
Proof. Suppose the assumptions hold and suppose $\int_{t_0}^{N} \varphi(t)\varphi(t)^T dt$ is singular, then there is a vector $v \in \mathbb{R}^3$ such that $\int_{t_0}^{N} (v^T \varphi(t))^2 dt = 0$ which implies $(v^T \varphi(t)) = 0$ $\forall t \in [t_0, N]$; then y

satisfies also the differential equation $\dot{y} = -ay + b$ for some a and b. Assuming first $d(t) = 0$ we have

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\theta_2 & -\theta_1 \end{pmatrix} x + \begin{pmatrix} 0 \\ \theta_3 \end{pmatrix}$$
$$x_2 = -ax_1 + b \ , \tag{7}$$

which leads to

$$x_2 = -\frac{\theta_2}{\theta_1 - a} x_1 + \frac{\theta_3}{\theta_1 - a}$$

and, with the second equation of (7), leads to

$$a = \frac{\theta_1 \mp \sqrt{\theta_1^2 - 4\theta_2}}{2} = -\lambda_{1,2}$$
$$b = \frac{\theta_3}{\theta_1 - a} = a\frac{\theta_3}{\theta_2} \tag{8}$$

and

$$x_2(t_0) = -ax_1(t_0) + b \ . \tag{9}$$

Substituting the values of a and b of (8) into (9) we find $x_2(t_0) = \lambda x_1(t_0) - \lambda\frac{\theta_3}{\theta_2}$ where $\lambda = \lambda_1, \ \lambda_2$ which contradicts the assumption. If $d(t) \neq 0$, equation (7) becomes

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\theta_2 & -\theta_1 \end{pmatrix} x + \begin{pmatrix} 0 \\ \theta_3 + d(t) \end{pmatrix}$$
$$x_2 = -ax_1 + b \ . \tag{10}$$

and by solving in function of x_1 we find $a^2x_1 - ab + b = \theta_2 x_1 + \theta_1 a x_1 - \theta_1 b + \theta_3 + d(t)$ which implies $x_1 = \alpha d(t) + \beta$, for suitable constants α and β. Using the second equality of (10) we find $\alpha\dot{d} = -a\alpha d(t) - a\beta + b$ which means that $d(t)$ has to satisfy a first order linear system which contadicts the assumption that $d(t)$ is white noise. $\square$

Lemma 2.2 Given the two dynamical systems

$$\ddot{y}_1(t) = \theta_1^T \varphi_1(t)$$
$$\ddot{y}_2(t) = \theta_2^T \varphi_2(t) \ , \tag{11}$$

$t \in [t_0, N]$, assume $\varphi_1(t_0) = \varphi_2(t_0)$, then $y_1(t) = y_2(t)$ $\forall t \in [t_0, N]$ iff $\theta_1 = \theta_2$.
Proof. ($\Leftarrow$) if $\theta_1 = \theta_2$ and $\varphi_1(t_0) = \varphi_2(t_0)$, by the uniqueness of solutions we have $y_1(t) = y_2(t)$.
Proof. ($\Rightarrow$) It is equivalent to prove that if $\theta_1 \neq \theta_2$, then $y_1(t) \neq y_2(t)$. Suppose instead that $y_1(t) = y_2(t)$ $\forall t \in [t_0, N]$, but $\theta_1 \neq \theta_2$ then we have $\dot{y}_1(t) = \dot{y}_2(t)$ and $\ddot{y}_1(t) = \ddot{y}_2(t)$, then using (11) we get $(\theta_1 - \theta_2)^T \varphi(t) = 0$ $\forall t \in [t_0, N]$, where $\varphi(t) = \varphi_1(t) = \varphi_2(t)$. This implies also that $((\theta_1 - \theta_2)^T \varphi(t))^2 = 0$ which integrated gives $\int_{t_0}^{N} ((\theta_1 - \theta_2)^T \varphi(t)^2) dt = 0$ which leads to

$$(\theta_1 - \theta_2)^T \int_{t_0}^{N} \varphi(t)\varphi(t)^T dt \ (\theta_1 - \theta_2) = 0$$

which in turn implies $(\theta_1 - \theta_2) \in Ker \ (\int_{t_0}^{N} \varphi(t)\varphi(t)^T dt)$ which contradicts Lemma 2.1. $\square$

We will focus on the case $\mathcal{M} = \{M^R, M^D\}$. Recalling Definition 2.1, we define the centers of the two sets $C^R = \{\theta_{R,k}\}$ and $C^D = \{\theta_{D,k}\}$ to be:

$$c_R = \frac{1}{|C^R|} \sum_{k=1}^{|C^R|} \theta_{R,k}$$

$$c_D = \frac{1}{|C^D|} \sum_{k=1}^{|C^D|} \theta_{D,K} \ ,$$

where $|C^j|$ denote the cardinality of the set C^j. From here on we assume that C^R and C^D are two balls in R^3 with centers in c_R and c_D and radii r_R and r_D respectively, i.e.,

$$C^R = B_{r_R}(c_R) \ , \qquad C^D = B_{r_D}(c_D) \ . \tag{12}$$

In this section we have proposed a definition for moveme, and on the basis of such a definition we showed the main properties that hold for moveme outputs. We have defined the particular model class M chosen, the set $\mathcal{M}$ of movemes and the sets C^R and C^D which parameterize the movemes.

3 Segmentation Problem

Given any signal $y(t)$, $t \in [t_0, N]$ which can be either non-segmentable or segmentable into two movemes outputs we would like to consider the problem of finding its actual segmentation $(s_1(t), s_2(t))_{n^*}$, in which $n^* = N$ for the non-segmentable case. We start by looking at the simplest case in which $y(t)$ is generated by a nominal system and then we extend the result to the case of a perturbed system. Given any signal $y(t)$ let $(s_1(t), s_2(t))_n$ be the current segmentation at time $n \in [t_0, N]$. Then we define the approximation error, e_a, as

$$0 \leq \ e_a(n) = \frac{1}{n - t_0} \int_{t_0}^{n} (s_1(t) - \hat{s}_1(t))^2 + \frac{1}{N-n} \int_{n}^{N} (s_2(t) - \hat{s}_2(t))^2 \ , \tag{13}$$

where $\hat{s}_1(t)$ and $\hat{s}_2(t)$ are the best representatives in the class M of $s_1(t)$ and $s_2(t)$ according to (4) and (5). Similarly, we define the parametric error, e_p, as

$$0 \leq e_p(n) = \|\hat{\theta}_1 - c_j\| + \|\hat{\theta}_2 - c_i\| \tag{14}$$

where $j = R$ if $\hat{\theta}_1 \in C^R$ or $j = D$ if $\hat{\theta}_1 \in C^D$, and analogously for i.

3.1 Nominal case

Consider the segmentation problem, with $i \in \{R, D\}$ and $j \in \{R, D\}$, for the nominal system

$$\ddot{y}^0(t) = \begin{cases} c_i^T \varphi^0(t) & t \in [t_0, n^*] \\ c_j^T \varphi^0(t) & t \in [n^*, N] \end{cases}$$
$$c_i^T \varphi^0(n^*) = c_j^T \varphi^0(n^*) \ . \tag{15}$$

Letting $(s_1^0(t), s_2^0(t))_n$ be the segmentation at time n of $y_0(t)$, output of system (15), we show that the quantities $e_a^0(n)$ and $e_p^0(n)$, computed as in (13) and (14) with $s_1 = s_1^0$, $s_2 = s_2^0$, $\hat{s}_1 = \hat{s}_1^0$, $\hat{s}_2 = \hat{s}_2^0$, $\hat{\theta}_1 = \hat{\theta}_1^0$, $\hat{\theta}_2 = \hat{\theta}_2^0$, admit their global minimizer at $n = n^*$, and that they satisfy first and second order necessary conditions for a minimizer in n^*. The problem of finding the actual segmentation point becomes a locally convex minimization problem.

Lemma 3.1 Let $e_a^0(n)$ and $e_p^0(n)$ be defined as in (13) and (14) for system (15), then $e_a^0(n) = 0$ iff $n = n^*$ and $e_p^0(n) = 0$ iff $n = n^*$.

Proof ($\Leftarrow$) Consider $(s_1^0(t), s_2^0(t))_n$ for (15) and suppose $n = n^*$, then $\ddot{s}_1^0(t) = c_i^T \varphi^0(t)$ and $\ddot{s}_s^0(t) = c_i^T \varphi^0(t)$. Solving (4) for $\hat{\theta}_1^0$ with $\ddot{y}^0(t) = c_i^T \varphi^0(t)$ and for $\hat{\theta}_2^0$ with $\ddot{y}^0(t) = c_j^T \varphi^0(t)$, we find:

$$(\int_{t_0}^{n^*} \varphi^0(t)\varphi^0(t)^T dt)(c_i - \hat{\theta}_1^0) = 0$$

$$(\int_{n^*}^{N} \varphi^0(t)\varphi^0(t)^T dt)(c_j - \hat{\theta}_2^0) = 0 \ .$$

Applying Lemma 2.1 we derive $\hat{\theta}_1^0 = c_i$ which by (5) implies $s_1^0(t) = \hat{s}_1^0(t)$, $t \in [t_0, t^*]$, and $\hat{\theta}_2^0 = c_j$ which implies $s_2^0(t) = \hat{s}_2^0(t)$, $t \in [t^*, N]$. Then by (13) and (14) we obtain the result.
($\Rightarrow$) Suppose $e_a^0(n) = 0$ and $e_p^0(n) = 0$, for $n < n^*$, we show that this leads to contradiction. Let $(s_1^0(t), s_2^0(t))_n$ be the segmentation at time n for system (15) then solving (4) for $\hat{\theta}_2^0$ we have

$$\hat{\theta}_2^0 = \text{argmin} \int_n^N (\ddot{y}^0(t) - \theta^T \varphi(t)^0)^2 dt$$

and computing the derivative with respect to θ with

$$\ddot{y}^0(t) = \begin{cases} c_i^T \varphi^0(t) & \in [n, n^*] \\ c_j^T \varphi^0(t) & t \in [n^*, N] \end{cases} \tag{16}$$

we find

$$\hat{\theta}_2^0(n) = \left(\int_n^N \varphi^0(t)\varphi^0(t)^T dt \right)^{-1} \left[\left(\int_n^{n^*} \varphi^0(t)\varphi^0(t)^T dt \right) c_i + \left(\int_{n^*}^N \varphi^0(t)\varphi^0(t)^T dt \right) c_j \right] \quad n < n^* \ . \tag{17}$$

Solving

$$\hat{\theta}_1^0 = \text{argmin} \int_{t_0}^n (\ddot{y}^0(t) - \theta^T \varphi^0(t))^2 dt$$

with

$$\ddot{y}^0(t) = c_i^T \varphi^0(t) \quad t \in [t_0, n] \tag{18}$$

we find

$$\hat{\theta}_1^0(n) = c_i \quad n < n^* \tag{19}$$

Recalling expression (14), in order to have $e_p^0(n) = 0$ we should then require (17) to be equal to c_D which leads to

$$\left(\int_n^N \varphi^0(t)\varphi^0(t)^T dt\right)^{-1} \left[\left(\int_n^{n^*} \varphi^0(t)\varphi^0(t)^T dt\right) c_i + \left(\int_{n^*}^N \varphi^0(t)\varphi^0(t)^T dt\right) c_j\right] - \left(\int_n^N \varphi^0(t)\varphi^0(t)^T dt\right)^{-1} \left[\left(\int_n^N \varphi^0(t)\varphi^0(t)^T dt\right) c_j\right] = 0 \;, \quad (20)$$

and splitting the last integral into two parts we obtain

$$\left(\int_n^N \varphi^0(t)\varphi^0(t)^T dt\right)^{-1} \left[\left(\int_n^{n^*} \varphi^0(t)\varphi^0(t)^T dt\right)\right] (c_i - c_j) = 0$$

that is satisfied if $(c_i - c_j) \in Ker(\int_n^{n^*} \varphi^0(t)\varphi^0(t)^T dt)$ which contradicts Lemma 2.1 according to which $Ker(\int_n^{n^*} \varphi^0(t)\varphi^0(t)^T dt)$ is empty for $n \neq n^*$. Therefore $e_p(n) = 0$ is not satisfied for $n < n^*$. As far as $e_a(n)$ is concerned, recalling expression (13), in order to have $e_a^0(n) = 0$ we need $s_2^0(t) = \hat{s}_2^0(t)$, where according to Definition 2.2 $s_2^0(t)$ given in (16) results to be segmentable, while $\hat{s}_2^0(t)$ is clearly a moveme output since we have by (5) that $\ddot{\hat{s}}_2^0 = (\hat{\theta}_2^0)^T \hat{\varphi}^0$. Then $s_2^0(t) = \hat{s}_2^0(t)$ implies in turn that $\hat{s}_2^0(t)$ is segmentable which contradicts proposition 2.1. The same analysis can be repeated for $n > n^*$, we report here for completeness the expressions for $\hat{\theta}_1(n)$ and $\hat{\theta}_2(n)$:

$$\hat{\theta}_1^0(n) = \left(\int_{t_0}^n \varphi^0(t)\varphi^0(t)^T dt\right)^{-1} \left[\left(\int_{n^*}^n \varphi^0(t)\varphi^0(t)^T dt\right) c_j + \left(\int_{t_0}^{n^*} \varphi^0(t)\varphi^0(t)^T dt\right) c_i\right] \quad n > n^* \;, \quad (21)$$

$$\hat{\theta}_2^0(n) = c_j \quad n > n^* \;. \quad (22)$$

□

Lemma 3.2 Let $e_a^0(n)$ and $e_p^0(n)$ be defined as in (13) and (14) for system (15), then $\frac{de_a^0(n)}{dn}|_{n=n^*} = 0$ and $\frac{de_p^0(n)}{dn}|_{n=n^*} = 0$; moreover there exists an interval I^0, with $n^* \in I^0$ in which $e_a^0(n)$ and $e_p^0(n)$ are C^1 functions.

Proof. We first show that $\hat{\theta}_1^0(n)$ and $\hat{\theta}_2^0(n)$ are C^1 functions of n. To start consider the expressions computed for $\hat{\theta}_2^0(n)$ and $\hat{\theta}_1^0(n)$:

$$\hat{\theta}_2(n) = \begin{cases} \left(\int_n^N \varphi^0(t)\varphi^0(t)^T dt\right)^{-1} \left[\left(\int_n^{n^*} \varphi^0(t)\varphi^0(t)^T dt\right) c_i + \left(\int_{n^*}^N \varphi^0(t)\varphi^0(t)^T dt\right) c_j\right] & n < n^* \\ c_j & n > n^* \end{cases} \quad (23)$$

and

$$\hat{\theta}_1(n) = \begin{cases} c_i & n < n^* \\ \left(\int_{t_0}^n \varphi^0(t)\varphi^0(t)^T dt\right)^{-1} \left[\left(\int_{n^*}^n \varphi^0(t)\varphi^0(t)^T dt\right) c_j + \left(\int_{t_0}^{n^*} \varphi^0(t)\varphi^0(t)^T dt\right) c_i\right] & n > n^* \;, \end{cases} \quad (24)$$

by applying standard classical analysis argument we conclude that since $\varphi^0(t)$ is a continuous function of t and by Lemma 2.1 $\int_n^N \varphi^0(t)\varphi^0(t)^T dt$ and $\int_{t_0}^n \varphi^0(t)\varphi^0(t)^T dt$ are nonsingular, then $\hat{\theta}_2^0(n)$ and $\hat{\theta}_1^0(n)$ are C^1 function for $n \in (t_0, n^*)$ and $n \in (n^*, N)$. To check if they are C^1 at n^* we compute if the limits

$$\lim_{n\to(n^*)^-} \hat{\theta}_2^0(n)$$

$$\lim_{n\to(n^*)^-} \frac{d\hat{\theta}_2^0(n)}{dn}$$

are equal to

$$\lim_{n\to(n^*)^+} \hat{\theta}_2^0(n)$$

$$\lim_{n\to(n^*)^+} \frac{d\hat{\theta}_2^0(n)}{dn}$$

respectively. By the first expression of (23) we find

$$\lim_{n\to(n^*)^-} \hat{\theta}_2^0(n) = \hat{\theta}_2^0(n^*) = c_j$$

and by the second expression of (24) we find

$$\lim_{n\to(n^*)^+} \hat{\theta}_1^0(n) = \hat{\theta}_1^0(n^*) = c_i \ ,$$

then $\hat{\theta}_1^0(n)$ and $\hat{\theta}_2^0(n)$ are continuous at n^*. For the derivative we find

$$\frac{d\hat{\theta}_2^0(n)}{dn} = \begin{cases} \left(\int_n^N \varphi^0(t)\varphi^0(t)^T dt\right)^{-1} (\varphi^0(n)\varphi^0(n)^T(\hat{\theta}_2^0(n) - c_i)), & n < n^* \\ 0 & n > n^* \end{cases} \tag{25}$$

and

$$\frac{d\hat{\theta}_1^0(n)}{dn} = \begin{cases} 0 & n < n^* \\ \left(\int_{t_0}^n \varphi^0(t)\varphi^0(t)^T dt\right)^{-1} (\varphi^0(n)\varphi^0(n)^T(\hat{\theta}_1^0(n) - c_j)), & n > n^* \ . \end{cases} \tag{26}$$

Then, taking the limits we find that $\lim_{n\to(n^*)^-} \frac{d\hat{\theta}_2^0(n)}{dn} = 0$ and $\lim_{n\to(n^*)^+} \frac{d\hat{\theta}_1^0(n)}{dn} = 0$ since, by (15), $c_i^T\varphi^0(n^*) = c_j^T\varphi^0(n^*)$. Then the derivatives are also continuous at n^* and their value is zero. Then

$$\frac{d\hat{\theta}_2^0(n)}{dn} = 0$$

$$\frac{d\hat{\theta}_1^0(n)}{dn} = 0 \tag{27}$$

$$\tag{28}$$

From expression (14), by computing the derivative with respect to n we find that $e_p(n)$ is also C^1 and by (28) $\frac{de_p^0(n)}{dn} = 0$. Note that to assert the continuity of $e_p^0(n)$ we need to assume that $\hat{\theta}_1 \in C^i$ and $\hat{\theta}_2 \in C^j$ when n varies around n^*; therefore to guarantee that this is the case we require

$$\|\hat{\theta}_2^0(n) - c_j\| < \frac{r_j}{2} \ , \qquad \|\hat{\theta}_1^0(n) - c_i\| < \frac{r_i}{2} \tag{29}$$

where $r_i = r_D$ if $i = D$ and $r_i = r_R$ if $i = R$ and the same for r_j. From expressions (24) and (23) we find the interval around n^* in which (29) is guaranteed; we call I^0 this interval and we find that

$$I^0 = \{n : |n^* - n| \leq \min\left(\frac{r_D \underline{\lambda}^-}{2\phi_M(\|c_R\| + \|c_D\|) + r_D\phi_M}, \frac{r_R \underline{\lambda}^+}{2\phi_M(\|c_R\| + \|c_D\|) + r_R\phi_M}\right)\} \tag{30}$$

where

$$\underline{\lambda}^- = \lambda_{\min}\left(\int_{n^*}^{N} \varphi^0(\varphi^0)^T dt\right)$$

$$\underline{\lambda}^+ = \lambda_{\min}\left(\int_{t_0}^{n^*} \varphi^0(\varphi^0)^T dt\right)$$

$$\phi_M = \max_t \|\varphi^0(t)\|^2 \ .$$

As far as $e_a(n)$ is concerned, expression (13) with

$$\ddot{\hat{s}}_1^0(t) = \hat{\theta}_1^0(n)^T \hat{\varphi}_1(t), \qquad t \leq n$$

$$\ddot{\hat{s}}_2^0(t) = \hat{\theta}_2^0(n)^T \hat{\varphi}_2(t), \qquad t \geq n$$

turns to be a continuous function of n since $\hat{s}_1^0(t, n)$ and $\hat{s}_2^0(t, n)$ coming out from the above differential equations are also continuous functions of n because $\hat{\theta}_1^0(n)$ and $\hat{\theta}_2^0(n)$ are so. Then

$$e_a(n) = \begin{cases} \frac{1}{N-n} \int_n^N (s_2^0(t) - \hat{s}_2^0(t, n))^2, & n \leq n^* \\ \frac{1}{n-t_0} \int_{t_0}^n (s_1^0(t) - \hat{s}_1^0(t, n))^2 & n \geq n^* \end{cases} \tag{31}$$

which is continuous at $n = n^*$ since in such a case $s_2^0(t) = \hat{s}_2^0(t, n^*)$ and $s_1^0(t) = \hat{s}_1^0(t, n^*)$. For the derivative we have

$$\frac{de_a^0(n)}{dn} = \begin{cases} -\frac{1}{(N-n)^2} \int_n^N (s_2^0(t) - \hat{s}_2^0(t, n))^2 dt + \frac{1}{N-n}(\int_n^N 2(s_2^0(t) - \hat{s}_2^0(t)) \frac{\partial \hat{s}_2^0(n)}{\partial n} dt) & \\ \qquad\qquad -\frac{1}{N-n}(s_2^0(n) - \hat{s}_2^0(n, n))^2 & n \leq n^* \\ -\frac{1}{(n-t_0)^2} \int_{t_0}^n (s_1^0(t) - \hat{s}_1^0(t, n))^2 dt + \frac{1}{t_0 - n}(\int_{t_0}^n 2(s_1^0(t) - \hat{s}_1^0(t)) \frac{\partial \hat{s}_1^0(n)}{\partial n} dt) & \\ \qquad\qquad -\frac{1}{t_0 - n}(s_1^0(n) - \hat{s}_1^0(n, n))^2 & n \geq n^* \end{cases} \tag{32}$$

from which we see that $\lim\limits_{n \to (n^*)^-} \frac{de_a^0(n)}{dn} = \lim\limits_{n \to (n^*)^+} \frac{de_a^0(n)}{dn} = 0$ since when $n = n^*$ the signals s_1^0 and s_2^0 coincide with their estimates. □

Therefore $e_a^0(n)$ and $e_p^0(n)$ are C^1 functions in I^0, both their derivatives are zero at $n = n^*$ and at such a point they have their global minimizer. It follows that they are locally convex and therefore they satisfy at $n = n^*$ the first and second order necessary conditions for a minimizer. The problem of finding the sctual segmentation point is then a locally convex minimization problem.

3.2 Perturbed case

We want to solve the segmentation problem for a signal $y(t)$, $t \in [t_0, N]$, which has been generated by a perturbed version of (15), namely by

$$\ddot{y}(t) = \begin{cases} (c_i + \delta v_1)^T \varphi(t) + d(t) & \in [t_0, n^*] \\ (c_j + \delta v_2)^T \varphi(t) + d(t) & \in [n^*, N] \end{cases} \tag{33}$$

where $\delta \in [0, \bar{\delta}]$, v_1 and v_2 are unit vectors, $d(t) \in [-\bar{d}, \bar{d}]$ $\forall t \in [t_0, N]$ is a bounded zero mean disturbance. Here δ represents the uncertainty on which particular models $M^j(\theta_j)$ and $M^i(\theta_i)$ in the classes M^j and M^i are generating the dynamics. We just know that $\theta_i = c_i + \delta v_1 \in B_\delta(c_i)$ and $\theta_j = c_j + \delta v_j \in B_\delta(c_j)$, with δ an unknown positive real for which we have an estimate of the set in which it lies. In the nominal case we showed that the segmentation problem can be solved by finding the minimizer for either $e_a^0(n)$ or $e_p^0(n)$, which is locally a convex minimization problem. With structure (33) we are not guaranteed anymore that $e_a(n)$ and $e_p(n)$ have a minimzer at $n = n^*$, we then look for a reasonable function to minimize, whose minimizer is still close (and we shortly will define what close means) to n^* under suitable assumptions. We could minimize just $e_a(n)$ given in (13), but as we will show soon the obtained estimates $\hat{\theta}_1$ and $\hat{\theta}_2$ are moved away from c_i and c_j by the presence of disturbance $d(t)$ and parameter uncertainty δ. In order to let $\hat{\theta}_1 \in C^i$ and $\hat{\theta}_2 \in C^j$ lie in the sets $B_\delta(c_i)$ and $B_\delta(c_j)$ in which θ_i and θ_j lie, we can either constrain $\hat{\theta}_1$ and $\hat{\theta}_2$ to lie into balls of radii δ around the centers c_i and c_j, either minimize $e_p(n)$ while minimizing $e_a(n)$. We choose the second option since in general we do not know *a priori* what i and j are, we then reformulate the problem of segmentation as an unconstrained optimization problem so to exploit the results of the previous section. In order to minimize to competitive quantities we choose as function to be minimized the product $e_a(n)e_p(n)$.

Lemma 3.3 Let $f^0(y)$ and $g^0(y)$, $y \in [y_0, y_M]$ be C^1 non negative functions which admit their global minimum at y^* with $f^0(y^*) = g^0(y^*) = 0$. Denote with I the smallest of the convexity intervals of $f^0(y)$ and $g^0(y)$ around y^*. Let $f(y)$ and $g(y)$ be perturbed versions such that:

$$\begin{aligned} f^0(y) - \varepsilon \leq f(y) \leq f^0(y) + \varepsilon \\ g^0(y) - \Delta \leq g(y) \leq g^0(y) + \Delta \end{aligned} \tag{34}$$

for all $y \in I^0 \subset I$ and $y^* \in I^0$. Then the minimizer $\bar{y} \in I^0$ of $f(y)g(y)$ is such that

$$(\bar{y} - y^*)^2 < \frac{(\varepsilon b + \Delta a) + \sqrt{(\varepsilon b + \Delta a)^2 + 8\varepsilon\Delta\bar{b}\bar{a}}}{2\bar{b}\bar{a}} \tag{35}$$

for suitable positive constants a, b, $\bar{a}$, $\bar{b}$.

Proof. For the function $f(y)g(y)$ we have

$$f^0(y)g^0(y) - \varepsilon g^0(y) - \Delta f^0(y) - \varepsilon\Delta \leq fg \leq f^0(y)g^0(y) + \varepsilon g^0(y) + \Delta f^0(y) + \varepsilon\Delta$$

Inequality (35) is found by considering that the minimizer $\bar{y} \in I^0$ has to satisfy $f^0(\bar{y})g^0(\bar{y}) - \varepsilon g^0(\bar{y}) - \Delta f^0(\bar{y}) - 2\varepsilon\Delta < 0$. We solve for $\bar{y}$ by letting $\bar{a}(y - y^*)^2 \leq f^0(y) \leq a(y - y^*)^2$ and

$\bar{b}(y-y^*)^2 \leq g^0(y) \leq b(y-y^*)^2$ for $y \in I^0$. $\square$

We proceed with the perturbation analysis: how $\hat{\theta}_1$ and $\hat{\theta}_2$ vary with respect to the nominal case (19) and (17), how the signals $s_1(t)$, $s_2(t)$, and their estimates $\hat{s}_1(t)$, $\hat{s}_2(t)$ vary with respect to the respective nominal signals $s_1^0(t)$, $s_2^0(t)$, $\hat{s}_1^0(t)$, $\hat{s}_2^0(t)$ introduced in the previous section.

Lemma 3.4 Let

$$\ddot{y}^0 = c^T\varphi^0(t) \tag{36}$$
$$\ddot{y} = (c+\delta v)^T\varphi(t) + d(t) \tag{37}$$

be the a nominal and the related perturbed system, with $0 \leq \delta \leq \bar{\delta}$ and $|d(t)| \leq \bar{d}\ \forall t$. If the nominal system and its perturbed version are asymtotically stable and $\varphi^0(t_0) = \varphi(t_0)$, then there exist two positive constants k_1 and k_2 such that

$$\|\varphi(t) - \varphi^0(t)\|^2 \leq k_1\bar{\delta}^2 + k_2\bar{d}^2 \ , \quad \forall t \geq t_0 \tag{38}$$

Proof. Consider the state space representations:

$$\dot{x}^0 = \begin{pmatrix} 0 & 1 \\ -c_2 & -c_1 \end{pmatrix} x^0 + \begin{pmatrix} 0 \\ c_3 \end{pmatrix} \tag{39}$$

and

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -(c_2+\delta v_2) & -(c_1+\delta v_1) \end{pmatrix} x + \begin{pmatrix} 0 \\ c_3+\delta v_3 \end{pmatrix} \tag{40}$$

where $x(t) = (y(t), \dot{y}(t))^T$ and $x^0(t) = (y^0(t), \dot{y}^0(t))^T$. Let $\tilde{x} = x^0 - x$, $A = \begin{pmatrix} 0 & 1 \\ -(c_2+\delta v_2) & -(c_1+\delta v_1) \end{pmatrix}$ and $A^0 = \begin{pmatrix} 0 & 1 \\ -c_2 & -c_1 \end{pmatrix}$ and construct the error system:

$$\dot{\tilde{x}} = \tilde{A}x^0 + \tilde{b} + A\tilde{x} + (0,1)^T d \ . \tag{41}$$

Consider the Lyapunov function $V = \frac{1}{2}\tilde{x}^T\tilde{x}$ for system (41), deriving it with respect to time we have

$$\dot{V} \leq -|\lambda|\|\tilde{x}\|^2 + \delta\|\tilde{x}\|\|x^0\| + \|\tilde{x}\|\delta + \|\tilde{x}\|d$$

where λ is the eigenvalue of A with smallest absolute value (it is negative since A is a.s. by assumption). Completing the squares we have:

$$\dot{V} \leq -(|\lambda| - \frac{1}{k})\|\tilde{x}\|^2 + \frac{1}{2}k\delta^2(\|x^0\|+1)^2 + \frac{k}{2}\|\bar{d}\|^2$$

in which k can be chosen so that $\lambda - \frac{1}{k} < 0$, choose $k = \frac{2}{\lambda}$, then

$$\dot{V} \leq -\frac{|\lambda|}{2}\|\tilde{x}\|^2 + \frac{|\lambda|}{4}\delta^2(\|x^0\|+1)^2 + \frac{|\lambda|}{4}\|\bar{d}\|^2$$

in which the second term is bounded by virtue of BIBO stability. Then if we assume that $\tilde{x}(t_0) = 0$, then $\tilde{x}^2$ will never leave the ball centered in the origin of radius $\frac{1}{2}\delta^2(\|x^0\| + 1)^2 + \frac{1}{2}\|\bar{d}\|^2$. Then, since $\|\varphi(t) - \varphi^0(t)\| = \| \begin{pmatrix} y(t) - y^0(t) \\ \dot{y}(t) - \dot{y}^0(t) \end{pmatrix} \|$, we have the thesis with $k_1 = \frac{1}{2}(\|x^0\| + 1)^2$ and $k_2 = \frac{1}{2}$.
□

The following lemma gives the relation between $e_p(n)$ and $e_p^0(n)$.

Lemma 3.5 Let $(s_1(t), s_2(t))_n$ be the segmentation at time n of the signal $y(t)$ generated by system (33); let $e_p(n)$ be the parametric error as defined in (14). Let $(s_1^0(t), s_2^0(t))_n$ be the segmentation at time n for $y^0(t)$ generated by the nominal system (15) and let $e_p^0(t)$ denote the parametric error for the nominal case. Then for $n \in I^0$ there exist functions $\bar{\delta}(r_D, r_R)$ and $\bar{d}(r_D, r_R)$ such that, if $\bar{\delta} < \bar{\delta}(r_D, r_R)$ and $\bar{d} < \bar{d}(r_D, r_R)$ we have

$$e_p^0(n) - \Delta \leq e_p(n) \leq e_p^0(n) + \Delta \tag{42}$$

with $\Delta = k_{\Delta,1}(\bar{d} + \bar{d}^2 + \bar{d}^4) + k_{\Delta,2}(\bar{\delta} + \bar{\delta}^2 + \bar{\delta}^3)$, $k_{\Delta,1}$ and $k_{\Delta,2}$ suitable positive constants, r_R and r_D given in (12).

Proof. We first find the expressions for $\hat{\theta}_1(n)$ and $\hat{\theta}_2(n)$ by solving (4) for $\hat{\theta}_1$ and $\hat{\theta}_2$ in analogy to what done in the nominal case. Then we find

$$\hat{\theta}_1 = \begin{cases} (c_i + \delta v_1) + \left[\int_{t_0}^{n} \varphi(t)\varphi(t)^T dt\right]^{-1} \int_{t_0}^{n} \varphi(t)d(t)dt & n < n^* \\ \left[\int_{t_0}^{n} \varphi(t)\varphi(t)^T dt\right]^{-1} \left(\int_{t_0}^{n^*} \varphi(t)\varphi(t)^T dt(c_i + \delta v_1)\right) + & \\ \left[\int_{t_0}^{n} \varphi(t)\varphi(t)^T dt\right]^{-1} \left(\int_{n^*}^{n} \varphi(t)\varphi(t)^T dt(c_j + \delta v_2) + \int_{t_0}^{n} \varphi(t)d(t)dt\right) & n > n^* \end{cases} \tag{43}$$

and

$$\hat{\theta}_2 = \begin{cases} \left[\int_{n}^{N} \varphi(t)\varphi(t)^T dt\right]^{-1} \left(\int_{n}^{n^*} \varphi(t)\varphi(t)^T dt(c_i + \delta v_1)\right) + & \\ \left[\int_{n}^{N} \varphi(t)\varphi(t)^T dt\right]^{-1} \left(\int_{n^*}^{N} \varphi(t)\varphi(t)^T dt(c_j + \delta v_2) + \int_{n}^{N} \varphi(t)d(t)dt\right) & n < n^* \\ (c_j + \delta v_2) + \left[\int_{n}^{N} \varphi(t)\varphi(t)^T dt\right]^{-1} \int_{n}^{N} \varphi(t)d(t)dt & n > n^* \end{cases} \tag{44}$$

To establish a relation between these expressions and the respective nominal case expressions (24) and (23), we substitute above $\varphi(t) = \varphi^0(t) + \tilde{\varphi}(t)$, with $\|\tilde{\varphi}\|$ given in (38), and find

$$\begin{aligned} &\hat{\theta}_1(n) = \hat{\theta}_1^0(n) + \tilde{\theta}_1(n)\ , \\ &\|\tilde{\theta}_1(n)\| \leq k_{11}(\bar{d} + \bar{d}^2 + \bar{d}^3) + k_{12}(\bar{\delta} + \bar{\delta}^2 + \bar{\delta}^3) \end{aligned} \tag{45}$$

for suitable constants k_{11} and k_{12} not depending on n.

$$\begin{aligned} &\hat{\theta}_2(n) = \hat{\theta}_2^0(n) + \tilde{\theta}_2(n) \\ &\|\tilde{\theta}_2(n)\| \leq k_{21}(\bar{d} + \bar{d}^2 + \bar{d}^3) + k_{22}(\bar{\delta} + \bar{\delta}^2 + \bar{\delta}^3) \end{aligned} \tag{46}$$

for opportune positive constants k_{21} and k_{22} not depending on n. In order to compute the relationship between $e_p(n)$ and $e_p^0(n)$ we assume that c_i and c_j of expression (14) are the same for perturbed and nominal case. By (29) in order to guarantee $\|\hat{\theta}_1(n)-c_i\| \le r_i$ and $\|\hat{\theta}_2(n)-c_j\| \le r_j$ for all $n \in I^0$ it is sufficient to ask $\|\tilde{\theta}_1\| < \frac{r_i}{2}$ and $\|\tilde{\theta}_2\| < \frac{r_j}{2}$ for all $n \in I^0$. Combining these two with the bounds for $\|\tilde{\theta}_1\|$ and $\|\tilde{\theta}_2\|$ given in (45) and (46) we find bouds for $\bar{\delta}$ and $\bar{d}$ which we call $\bar{\delta}(r_D, r_R)$ and $\bar{d}(r_D, r_R)$. Finally we obtain $e_p = \|\hat{\theta}_2-c_j\|+\|\hat{\theta}_1-c_i\| = \|\theta_2^0+\tilde{\theta}_2-c_j\|+\|\theta_1^0+\tilde{\theta}_1-c_i\| \le e_p^0 + \|\tilde{\theta}_1\| + \|\tilde{\theta}_2\|$ which gives the result with $k_{\Delta,1} = k_{11} + k_{21}$ and $k_{\Delta,2} = k_{12} + k_{22}$. $\square$

The following lemma gives the relation between $e_a(n)$ and $e_a^0(n)$.

Lemma 3.6 let $(s_1(t), s_2(t))_n$ be the segmentation at time n of $y(t)$ generated by (33) and let $e_a(n)$ be the approximation error as defined in (13). Let $(s_1^0(t), s_2^0(t))_n$ be the segmentation at time n for signal $y^0(t)$ generated by (15) and let $e_a^0(n)$ be the nominal approximation error. Then

$$e_a^0(n) - \varepsilon \le e_a(n) \le e_a^0(n) + \varepsilon \tag{47}$$

with $\varepsilon = k_{\varepsilon,1}(\bar{d} + \bar{d}^2 + \bar{d}^4 + \bar{d}^8) + k_{\varepsilon,2}(\bar{\delta} + \bar{\delta}^2 + \bar{\delta}^3 + \bar{\delta}^4 + \bar{\delta}^6)$, for $k_{\varepsilon,1}$ and $k_{\varepsilon,2}$ opportune positive constants.

Proof. We proceed by finding a relationship between $s_1(t)$, $s_2(t)$, $\hat{s}_1(t)$, $\hat{s}_2(t)$ and the respective nominal quantities $s_1^0(t)$, $s_2^0(t)$, $\hat{s}_1^0(t)$, $\hat{s}_2^0(t)$. Suppose $n \le n^*$ ($n > n^*$ is equivalent). Then segments $s_1(t)$ and $s_1^0(t)$ are generated respectively by

$$\begin{aligned} \ddot{s}_1(t) &= (c_i + \delta v_1)^T \varphi + d(t), \quad t \in [t_0, n] && (48)\\ \ddot{s}_1^0 &= c_i^T \varphi^0, \quad t \in [t_0, n]\ . && (49) \end{aligned}$$

Applying Lemma 3.4 to the nominal and related perturbed systems given in (49) and (48), we get

$$s_1(t) = s_1^0(t) + \tilde{s}_1(t)\ , \qquad |\tilde{s}_1(t)|^2 \le k_{11}^{s-}\bar{\delta}^2 + k_{12}^{s-}\bar{d}^2\ ,$$

where the superscript "$-$" indicates the case $n < n^*$. For $s_2(t)$ and $s_2^0(t)$ generated respectively by

$$\ddot{s}_2(t) = \begin{cases} (c_i + \delta v_1)^T \varphi(t) + d(t) & t \in [n, n^*] \\ (c_j + \delta v_2)^T \varphi(t) + d(t) & t \in [n^*, N] \end{cases} \tag{50}$$

$$\ddot{s}_2^0 = \begin{cases} c_i^T \varphi^0(t) & t \in [n, n^*] \\ c_j^T \varphi^0(t) & t \in [t^*, N] \end{cases} \tag{51}$$

Applying Lemma 3.4 to nominal and perturbed systems given in (51) and (50) we find

$$s_2(t) = s_2^0(t) + \tilde{s}_2(t), \qquad |\tilde{s}_2(t)|^2 \le k_{21}^{s-}\bar{\delta}^2 + k_{22}^{s-}\bar{d}^2\ .$$

For the estimates we have that $\hat{s}_1$ and $\hat{s}_1^0$ are generated by

$$\ddot{\hat{s}}_1 = (\hat{\theta}_1^0 + \tilde{\theta}_1)^T \hat{\varphi}(t) \quad t \in [t_0, n] \tag{52}$$

$$\ddot{\hat{s}}_1^0 = \hat{\theta}_1^0 \hat{\varphi}^0(t), \quad t \in [t_0, n]\ . \tag{53}$$

Applying Lemma 3.4 to (52) and (53), with $\delta = \|\tilde{\theta}_1\|$ and $d = 0$, we find

$$\hat{s}_1(t) = \hat{s}_1^0(t) + \tilde{\hat{s}}_1(t), \qquad \|\tilde{\hat{s}}_1(t)|^2 \leq k_3^{s-}\|\tilde{\theta}_1\|^2 \ .$$

Applying the same argument for $\hat{s}_2$ and $\hat{s}_2^0$ we obtain

$$\hat{s}_2(t) = \hat{s}_2^0(t) + \tilde{\hat{s}}_2(t), \qquad \|\tilde{\hat{s}}_2(t)|^2 \leq k_4^{s-}\|\tilde{\theta}_2\|^2 \ .$$

We can apply the same arguments in the symmetric case $n > n^*$ to obtain the same relations obtained above with the superscript "+". Therefore we finally obtain for all $n \in (t_0, N)$

$$\begin{array}{rcll} s_1(t) & = & s_1^0(t) + \tilde{s}_1(t) \ , & |\tilde{s}_1(t)|^2 \leq k_{11}^s\bar{\delta}^2 + k_{12}^s\bar{d}^2 \ , \\ s_2(t) & = & s_2^0(t) + \tilde{s}_2(t), & |\tilde{s}_2(t)|^2 \leq k_{21}^s\bar{\delta}^2 + k_{22}^s\bar{d}^2 \ , \\ \hat{s}_1(t) & = & \hat{s}_1^0(t) + \tilde{\hat{s}}_1(t), & |\tilde{\hat{s}}_1(t)|^2 \leq k_3^s\|\tilde{\theta}_1\|^2 \ , \\ \hat{s}_2(t) & = & \hat{s}_2^0(t) + \tilde{\hat{s}}_2(t), & |\tilde{\hat{s}}_2(t)|^2 \leq k_4^s\|\tilde{\theta}_2\|^2 \ . \end{array} \tag{54}$$

Finally rewriting $e_a(n)$ as

$$e_a(n) = \frac{1}{n-t_0}\int_{t_0}^{n}((s_1^0 - \hat{s}_1^0) + (\hat{s}_1 - \hat{s}_1^0 + s_1 - s_1^0))^2 dt + \frac{1}{N-n}\int_{n}^{N}((s_2^0 - \hat{s}_2^0) + (\hat{s}_2 - \hat{s}_2^0 + s_2 - s_2^0))^2 dt$$

and developping the suares we find

$$e_a(n) = e_a^0(n) + \frac{1}{n-t_0}\int_{t_0}^{n}(2(s_1^0 - \hat{s}_1^0)(\tilde{\hat{s}}_1 + \tilde{s}_1) + (\tilde{\hat{s}}_1 + \tilde{s}_1)^2)dt + \frac{1}{N-n}\int_{n}^{N}(2(s_2^0 - \hat{s}_2^0)(\tilde{\hat{s}}_2 + \tilde{s}_2) + (\tilde{\hat{s}}_2 + \tilde{s}_2)^2)dt \ .$$

Using (54) with (45) and (46) we find (47). □

Then we can combine results 3.5, Lemma 3.6 and Lemma 3.3 to derive the main result:

Theorem 3.1 *Given system (33), there exist functions* $\bar{\delta}(r_R, r_D)$ *and* $\bar{d}(r_R, r_D)$ *such that if* $\bar{\delta} < \bar{\delta}(r_R, r_D)$ *and* $\bar{d} < \bar{d}(r_R, r_D)$ *then the solution of the segmentation problem for* $y(t)$*, found by minimizing the product* $e_a(n)e_p(n)$ *over* $n \in I^0$*, is the real* $\bar{n}$ *which satisfies*

$$(\bar{n} - n^*)^2 \leq \frac{(\varepsilon b + \Delta a) + \sqrt{(\varepsilon b + \Delta a)^2 + 8\varepsilon\Delta\bar{b}\bar{a}}}{2\bar{b}\bar{a}} \tag{55}$$

with $\Delta = k_{\Delta,1}(\bar{d} + \bar{d}^2 + \bar{d}^4) + k_{\Delta,2}(\bar{\delta} + \bar{\delta}^2 + \bar{\delta}^3)$ *and* $\varepsilon = k_{\varepsilon,1}(\bar{d} + \bar{d}^2 + \bar{d}^4 + \bar{d}^8) + k_{\varepsilon,2}(\bar{\delta} + \bar{\delta}^2 + \bar{\delta}^3 + \bar{\delta}^4 + \bar{\delta}^6)$*,* a*,* b*,* $\bar{a}$*,* $\bar{b}$ *positive constants,* r_D *and* r_R *defined in (12).*

Proof.
It is a direct consequence of Lemma 3.5, Lemma 3.6 and Lemma 3.3. Infact Lemma 3.5 and Lemma 3.6 show that the functions $e_a(n)$ and $e_a^0(n)$, $e_p(n)$ and $e_p^0(n)$ satisfy the hypothesis of Lemma 3.5 which can be applied with $n = y$,$e_a = f$, $e_a^0 = f^0$, $e_p = g$, $e_p^0 = g^0$.

4 Appendix

4.1 Exact computation of bounds (45) and (46)

We consider the computation for $\hat{\theta}_1$ since for $\hat{\theta}_2$ the procedure is the same. Given relation (38) we can write

$$\int_a^b \varphi(t)\varphi(t)^T dt = \int_a^b \varphi^0(t)\varphi^0(t)^T dt + \int_a^b \varphi^0(t)\tilde{\varphi}(t)^T dt + \int_a^b \tilde{\varphi}(t)\varphi^0(t)^T dt + \int_a^b \tilde{\varphi}(t)\tilde{\varphi}(t)^T dt \ ; \quad (56)$$

letting $\Phi_a^b = \int_a^b \varphi(t)\varphi(t)^T dt$, $\tilde{\Phi}_a^b = \int_a^b \varphi^0(t)\tilde{\varphi}(t)^T dt + \int_a^b \tilde{\varphi}(t)\varphi^0(t)^T dt + \int_a^b \tilde{\varphi}(t)\tilde{\varphi}(t)^T dt$ and $(\Phi^0)_a^b = \int_a^b \varphi^0(t)\varphi^0(t)^T dt$ we rewrite (56) as

$$\Phi_a^b = (\Phi^0)_a^b + \tilde{\Phi}_a^b \quad (57)$$

and by (38) $\tilde{\Phi}$ is such that:

$$\|\tilde{\Phi}_a^b v\| \leq (b-a)\sup_t(2\|\varphi^0(t)\|\|\tilde{\varphi}(t)\| + \|\tilde{\varphi}(t)\|^2) \leq (b-a)(2\|\bar{\varphi^0}\|(\sqrt{k_1}\bar{d} + \sqrt{k_2}\bar{\delta}) + k_1\bar{d}^2 + k_2\bar{\delta}^2) \quad (58)$$

Considering expressions (24) and (43), by adding and subtracting in (43) expressions (24) we can rewrite $\hat{\theta}_1$ as

$$\hat{\theta}_1 = \begin{cases} \hat{\theta}_1^0(n) + \delta v_1 + [\Phi_{t_0}^n]^{-1} \int_{t_0}^n \varphi(t)d(t)dt & n < n^* \\ \hat{\theta}_1^0(n) + [(\Phi_{t_0}^n)^0 + \tilde{\Phi}_{t_0}^n]^{-1}[(\Phi_{t_0}^n)^0]^{-1}\left(-\tilde{\Phi}_{t_0}^n((\Phi_{t_0}^{n^*})^0 c_i + (\Phi_{n^*}^n)^0 c_j)\right) & \\ +[(\Phi_{t_0}^n)^0 + \tilde{\Phi}_{t_0}^n]^{-1}[(\Phi_{t_0}^n)^0]^{-1}\left((\Phi_{t_0}^n)^0(\tilde{\Phi}_{t_0}^{n^*} c_i + \tilde{\Phi}_{n^*}^n c_j + \delta\Phi_{t_0}^{n^*} v_1 + \delta\Phi_{n^*}^n v_2 + \int_{t_0}^n \varphi(t)d(t)dt)\right) & n > n^* \end{cases} \quad (59)$$

and using (57) and (58) we obtain (45). The same procedure holds for θ_2.

5 The Segmentation Algorithm

The actual segmentation algorithm, implemented in MATLAB 6.0 and then run on real data, minimizes the function $E(n) = e_p(n)e_a(n)(\alpha + e_d(n))$, in which we introduced the additional term $e_d(n)$ defined as $e_d(n) = |\hat{s}_1(n) - \hat{s}_2(n)| + |\hat{s}_2(N) - y(N)|$. This term accounts for the discontinuity of the estimate at $t = n$ and the discontinuity at $t = N$; both terms are shown in figure 1. The constant α is arbitrary positive. The general structure of the minimization algorithm is

- specify a guess of solution n_0
- while $\frac{dE(n)}{dn} < 0$
  $n_{k+1} = n_k + \Delta n_k$
- stop

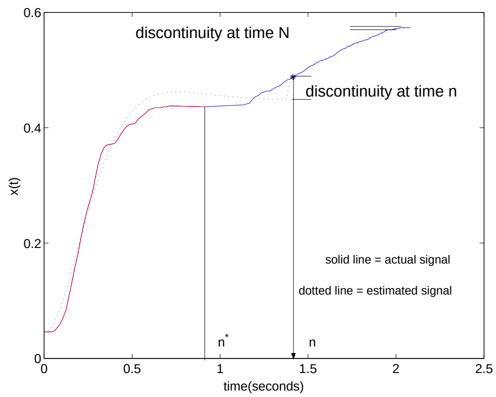

Figure 1: **Discontinuity terms**

Typically one would choose $\Delta n_k = -\frac{dE(n)}{dn}|_{n_k}\eta_k$, with η_k chosen according to the backtraking technique for example (see (Nash and Sofer, 1996)). It is verifiable in a few steps that $\frac{dE(n)}{dn} < 0$ is equivalent to

$$\frac{e_p'(n)}{e_p(n)+\beta} + \frac{e_a'(n)}{e_a(n)+\gamma} + \frac{e_d'(n)}{\alpha + e_d(n)} < 0 \ , \tag{60}$$

where β and γ have been introduced to avoid having zero denominators and they are positive arbitrary constants. This form is nice because it gives evidence that in the minimization process we are looking at each step for a global percentage decrease of the functions $e_a(n)$, $e_p(n)$ and $e_d(n)$. Since in practice we have just a sampled version of the functions $e_a(n)$, $e_p(n)$ and $e_d(n)$, where n is a non-negative integer, then the derivatives need to be repaced by finite differences. It can be shown in few steps that (60) becomes

$$\frac{e_p(n)-e_p(n-1)}{\frac{e_p(n)+e_p(n-1)}{2}+\beta} + \frac{e_a(n)-e_a(n-1)}{\frac{e_a(n)+e_a(n-1)}{2}+\gamma} + \frac{e_d(n)-e_d(n-1)}{\alpha+\frac{e_d(n)+e_d(n-1)}{2}} < 0 \tag{61}$$

and at each step $n \leftarrow n+1$. To avoid to end up into local minima the structure of the minimization algorithm has been transformed to

- let $n^o = n_0$ be the initial guess of the minimizer
- for n=1,2,3,... check if n^o is not the optimum, i.e. if

$$\frac{e_p(n)-e_p(n^o)}{\frac{e_p(n)+e_p(n^o)}{2}+\beta} + \frac{e_a(n)-e_a(n^o)}{\frac{e_a(n)+e_a(n^o)}{2}+\gamma} + \frac{e_d(n)-e_d(n^o)}{\alpha+\frac{e_d(n)+e_d(n^o)}{2}} < 0 \tag{62}$$

  update the new minimizer n^o to n.
- set $\bar{n}$ to n^o.

Definition 2.2 clearly establishes that the output generated by system (15) is segmentable. The same definition does not apply to the output of system (33). Then let $y(t)$, $t \in [t_0, N]$, be the output of system (33), we establish that $y(t)$ is segmentable if

$$e_p(N) > 0.5 e_p(\bar{n}) \ , \tag{63}$$

where $\bar{n}$ is the minimizer found with the process described above. Since we also have also a sampled version of the signal $y(t)$ instead of representation (3), we use a discrete time LTI representation, see (Ljung, 1999),

$$\begin{aligned} y(t) &= \theta^T \varphi(t), \\ \varphi(t)^T &= (-y(t-1), -y(t-2), 1(t-1)) \ , \end{aligned} \tag{64}$$

so that we avoid measuring the acceleration and we just measure the output $y(t)$ itself.

6 Experiments

To validate our starting hypothesis that there may exist a set of atomic motions, we registered the trajectory of the hand in the plane when the person was asked to accomplish two different tasks which at a first glance seemed to be indistinguishable. We chose the two actions “reach a point” and “draw a straight line”.

6.1 Dataset

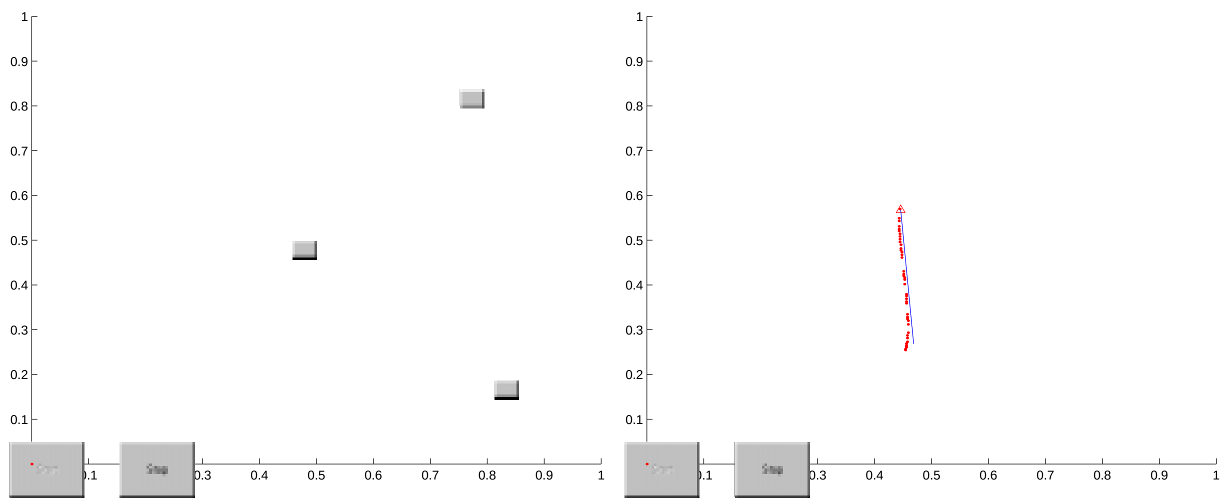

Figure 2: **Screen setup: on the left the setup for a reach sequence, on the right the set up for a draw sequence which shows also the trace acquired in the experiment**

We carried out our experiments on trajectories captured on four human subjects. Two simple videogames were implemented in Matlab for this purpose on a commercial PC running Windows NT. The screen of the PC measured 1600×1200 pixels^2 and the working window was 800×600 pixels^2. The position of the mouse cursor was tracked from Matlab using the function “get(gca,’Current Point’)” which sampled the data at approximatively 100Hz if the mouse in the working area was mooving. In the first “point-and-click” game a “sequence” was initiated when three about 20×20 pixel boxes appeared at random positions on the screen. The user, starting from a base location, had to point and click inside each of the boxes and then click inside a box indicating the base to terminate the sequence. In the second “point-and-draw” game a “sequence” was initiated when a straight line with marked extreme appeared at random positions and inclinations. The user starting again from a base location had to point the marked extreme of the line and then trace a new overlapping line, then click inside a box indicating the base the end of the sequence. This second game had an other option in which two connected lines with random inclinations and positions were appearing instead of one. The users were allowed to practice for approximatively 3 sequences for each game so to carry out each task in a natural way. In total about 70 sequences for each task were captured for each of the four subjects. The average length of a point-and-click sequence was 157 points and for a point-and-draw sequence was 182 points.

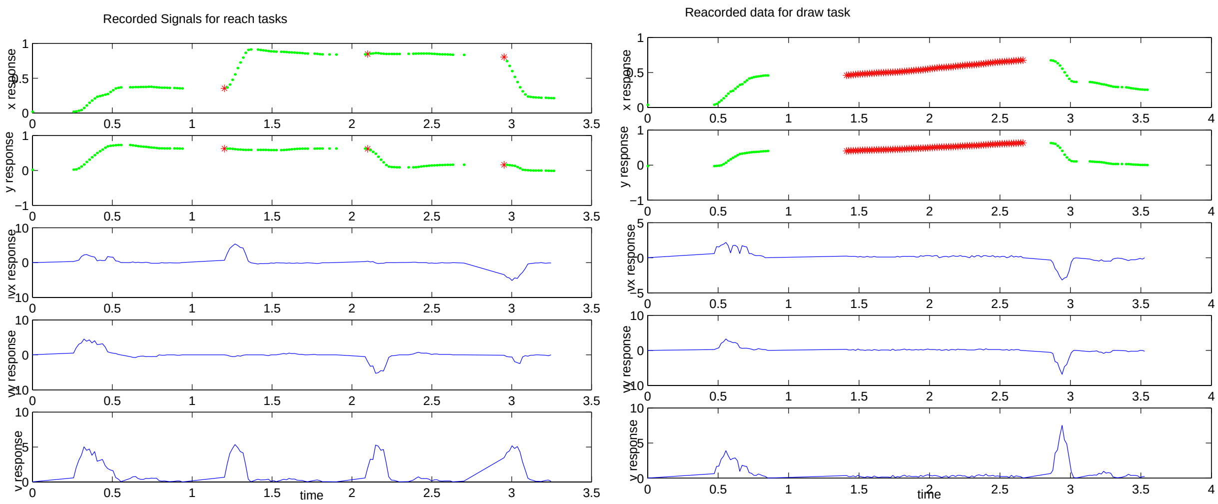

Figure 3: **Examples of a x reach raw sequence (on the left) and of a draw raw sequence (on the right)**

Figure 2 shows the screen setup. In Figure 3 we show the raw data acquired with the described experiments. By looking at the velocity profiles of a reach and a draw motion, we notice how these motions are different: for a reach motion we have a kind of bell shape profile, while for the draw motion the velocity has almost a trapezoidal profile. The same kind of profiles are obtained in all the experiments and across all the subjects. This distinction suggests that from a dynamical point of view these two movements are generated by clearly different dynamics.

6.2 Classification Problem

For each reach sample and for each draw sample we used (64) to model the dynamics on y and x axis and (4) (with the mentioned modifications for the discrete case) to estimate the reach and draw parameters $\theta_R = ((\theta_R^x)^T, (\theta_R^y)^T))^T$ and $\theta_D = ((\theta_D^x)^T, (\theta_D^y)^T))^T$, where the superscript denotes the axis (y or x) whose dynamics was considered for the parameter estimation. The first problem we deal with is the one of correctly classifying a new reach or draw sample as reach or draw based on the dynamical parameters estimated for a training set of reach and draw samples. Then given a training set of reach/draw parameters, which we call $\tilde{\Theta}^R = \{\theta_{R,j}\}$ and $\tilde{\Theta}^D = \{\theta_{D,j}\}$ respectively, we train a classifier to distinguish between the two sets of parameters. We use the Fisher classifier (see (Bishop, 1995)) for x and y parameters separately which projects the parameters along the first two Fisher linear discriminants; in other words we find based on the training sets a linear transformation which maps the above sets into the sets $\tilde{C}^R$ and $\tilde{C}^D$ respectively. The distribution of x reach and draw parameters for a typical user is shown in Fig. 4 and it is clear that they separate. Then we train a linear neural network with signum activating function to classify the reach and draw sets both for y and x directions. We chose the data of different people for training and testing since in general it is likely that we have to classify the actions of people who never partecipated in previous experiments. Letting the above defined sets denote the x and y training sets, we obtained a training error of 6/116 (5.17%) and test error of 5/76 (6.5%) for

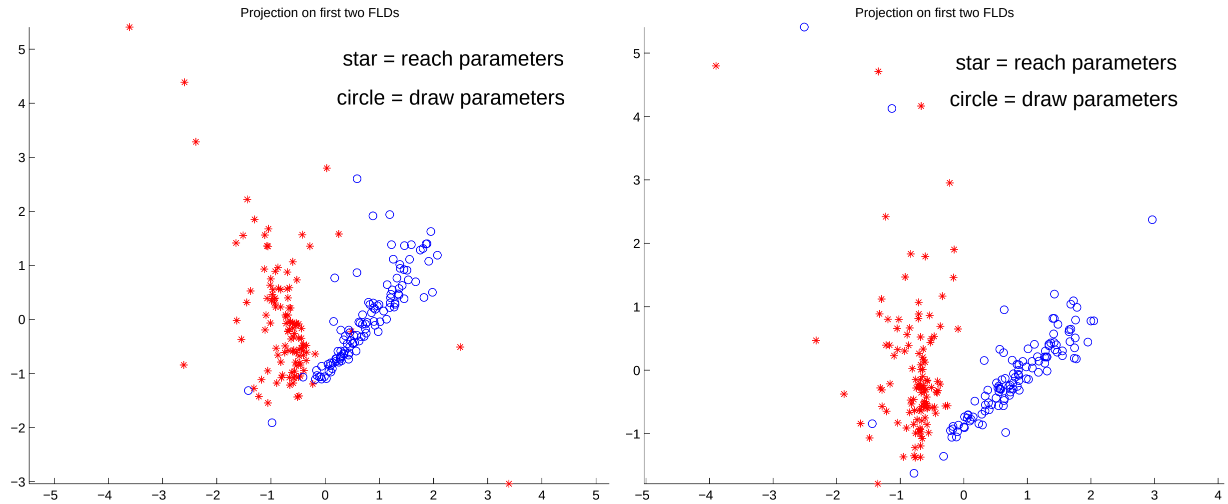

Figure 4: **Reach and draw parameters for the x dynamics (on the left) and the y dynamics (on the right)**

the x parameters, while a training error of 4/116 (3.4%) and a test error of 4/76 (5.26%) for the y parameters.
Then excluding from the sets $\tilde{C}^R$ and $\tilde{C}^D$ the parameters that were missclassified in this process we obtain the sets C^R and C^D which are linearly separable and then satisfy Definition 2.1. These sets parametrize two sets of LTI dynamical systems $M^R \subset M$ and $M^D \subset M$ which according to Definition 2.1 are dynamical independent sets of models. Then we have found two *movemes*, the reach and draw movemes, whose output is the synthetic reproduction of the time sequence of a reach action and a draw action.

Remark 6.1 Thee separability brtween reach and draw parameters is a finding which is consistent across subjects. We infact did some experiments in which we trained the Fisher classifier to distinguish between reach (either draw) parameters of two different people and the result is shown in Figure 5. It is clear that the parameters of the same motion (reach or draw) of different people completely overlapp. This means that the dynamics of reach and draw motions here analyzed have intrinsic characteristics which do not depend on the particular subject that is accomplishing the action.

7 Segmentation Algorithm Results

Table 1: Confusion Matrix

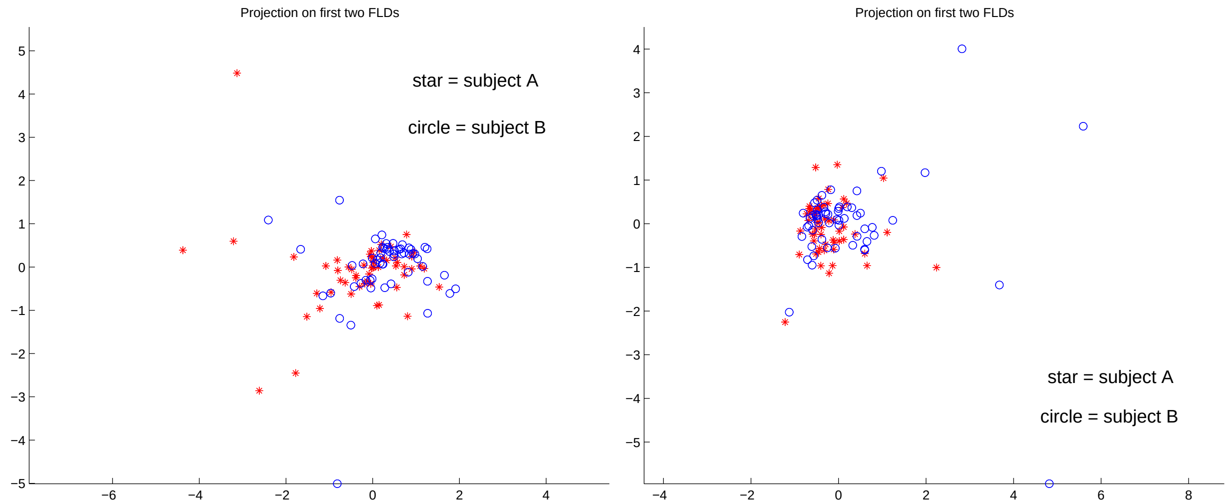

Figure 5: **Consistency across subjects: reach (on the left) and draw (on the right) parameters for two different subjects**

	Predicted					
Actual	R	D	R/D	R/R	D/D	D/R
R	94	2	4	11	0	4
D	0	71	1	0	5	0
R/D	3	0	99	0	0	0
R/R	22	0	13	75	0	5
D/D	0	18	0	0	71	1
D/R	1	2	0	1	5	91

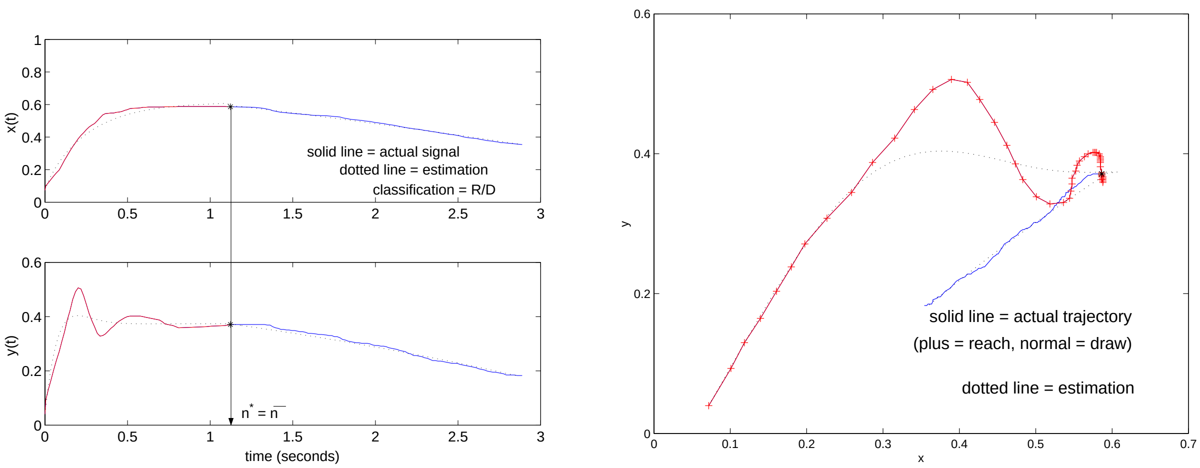

Figure 6: **Segmentation result in a R/D case: the signal in the time (on the left) and the trajectory in xy plane (on the right)**

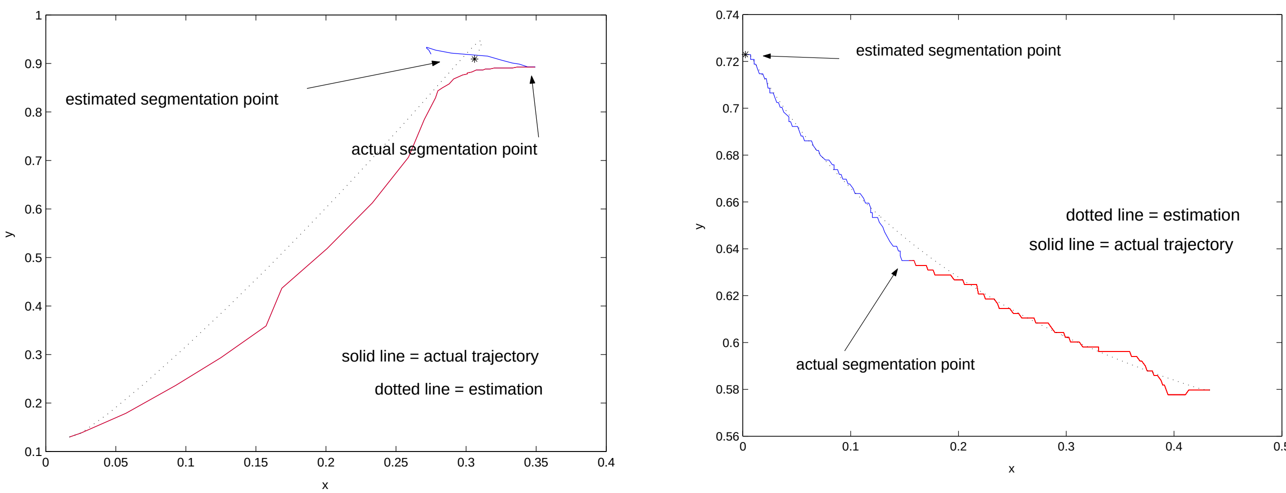

Figure 7: **Critical sequences: R/R (on the left) and D/D (on the right)**

We ran the segmentation algorithm with check (62) and segmentability check (63) on the data. The classification algorithm is a subroutine of the segmentation algorithm and it is necessary to compute the proper parametric errors (14); the final outputs of the segmentation algorithm are the estimated segmentation point n and the classification of the segments found. Therefore the answer of the algorihm is correct if it has provided not only the right segmentation point, but also the right classification. In the code the possibility of recognising when the hand is not mooving has been included: the periods in which nothing happens can be identified as a pause in the resulting segmentation when it produces smaller values of the cost E. In the segmentation process we compute the quantities $e_a^x(n)$, $e_a^y(n)$, $e_p^x(n)$, $e_p^y(n)$ separately for x and y channels and then $e_a(n) = e_a^x(n) + e_a^y(n)$ and $e_p(n) = e_p^x(n) + e_p^y(n)$. The resulting errors (mis-segmentation or correct segmentation but wrong classification) are reported in the confusion matrix. The best results are obtained for R/D and D/R sequences for which in Fig. 6 we report an example, while the worst are for R/R and D/D sequences. The reason is that several of the R/R or D/D sequences really looked like just one R or D movement: in several cases the D/D sequence was performed with two lines that were almost aligned and the R/R sequence was performed reaching points which were very close to each others. In these cases the algorithm improperly classify the results. Two examples of these cases are shown in Figure 7.

8 Conclusions

We have proposed a dynamical formulation of movemes. We restricted our attention to two dimensional movements and showed that there exist two movemes that have dynamical characteristics which are sufficient to distingush between them. The experiments also showed that the clusters in parameter space are not subject dependent. The segmentation algorithm was tested on about 600 samples of composed and simple actions and it gave approximatively 90% accuracy. Our analysis of second order LTI systems can also be extended to more complex dynamical systems. In future we plan to aquire more data and look for other primitives of motion to add in the alphabet; we will also consider the case in which the action to be segmented is segmentable in more than two parts and generalize to the case of three dimensional motion.

Acknowledgements

Funded in part by the NSF Engineering Research Center for Neuromorphic Systems Engineering (CNSE) at Caltech (NSF9402726), and by an NSF National Young Investigator Award to PP (NSF9457618). We thank all the people who partecipated in the experiments.

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