Fundamental Limitations of Feedback Control with Communication Constraints

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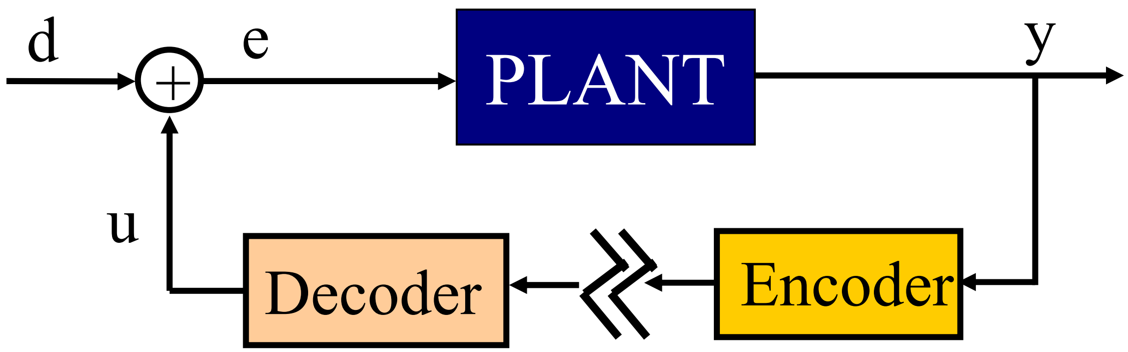
A Simple Network Problem



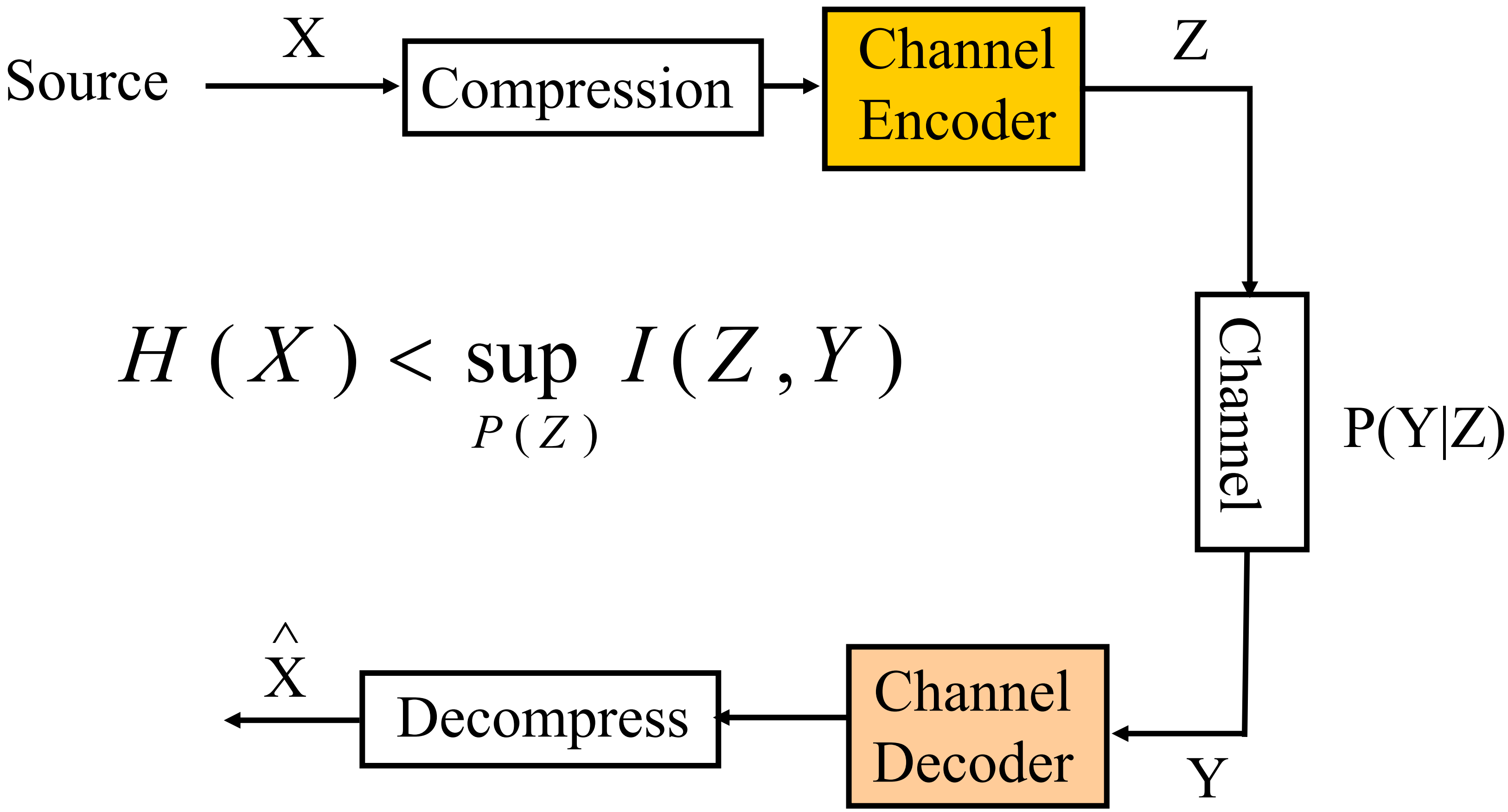
Objective:

- Stability and disturbance rejection

Requires:

- Stability ⇔ Channel sends reliable estimate of the state
- Performance ⇔ Channel has to send info about disturbance

Information Transmission: Shannon



$$H(X) < \sup_{P(Z)} I(Z,Y)$$

Anytime Capacity (Sahai 2000)

Shannon is not enough!

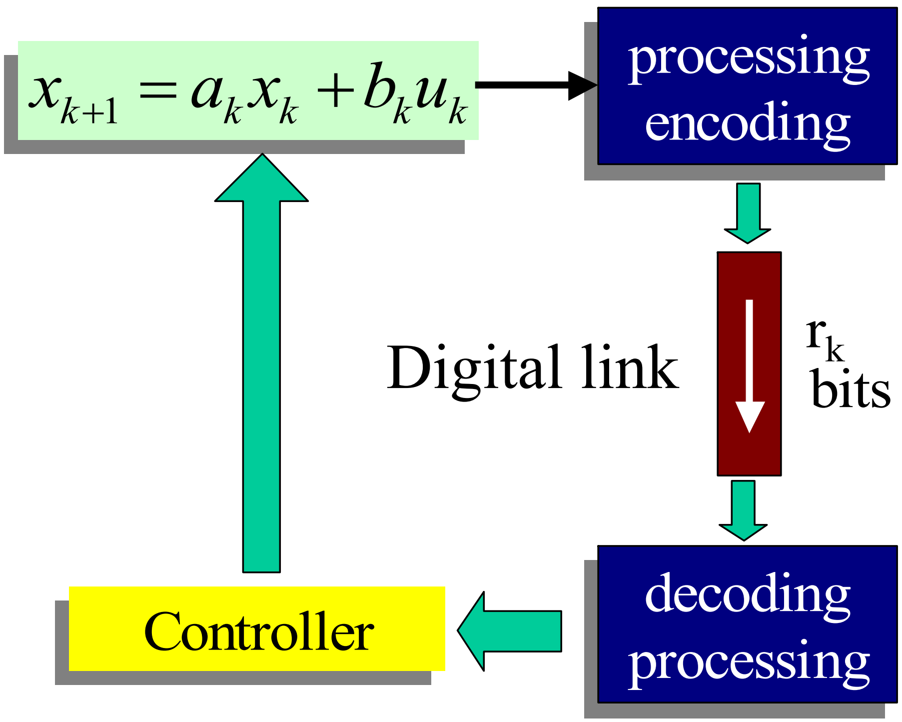
- Asymptotic theory
- Delays are critical for stability.
- Blocking requires stabilization at a slower rate.
- Decrease in probability of error has to be accounted for as a function of block size.

$$C_{anytime}(\alpha) = \text{ supremum of rates achievable with } P_e(N) \propto e^{-N\alpha}$$

- Moment stability can be equivalently expressed using $C_{anytime}$
- Difficult to compute except for special cases.

Problem Formulation

Definition: A feedback scheme is the collection of an encoder, a decoder and a controller.



Consider the following:

- a_k is a stochastic process satisfying:

$$\log |a_k| = R + \delta_a(k)$$

$\delta_a(k)$: zero mean

- r_k is a bounded stochastic process satisfying:

$$r_k = C + \delta_r(k)$$

$\delta_r(k)$: zero mean

Previous Work

- Estimation of unstable processes (Li & Wong 1996)
- Stabilization of LTI systems under information constraints (almost-sure stability) (Tatikonda & Mitter, 2002)

$$C > \max\{0, \log(|a|)\}$$

- Moment stability for LTI systems and stochastic channels: Anytime capacity, (Sahai &Mitter, 2000)
- Study of decentralized Schemes (Yuksel, Basar, 2003)
- Moment stability and binary stochastic channels. (Jain, Simsek, Varayia, 2003)
- Relations between LQ controllers and channel capacity for Gaussian Channels (Elia)

Almost Sure Stability

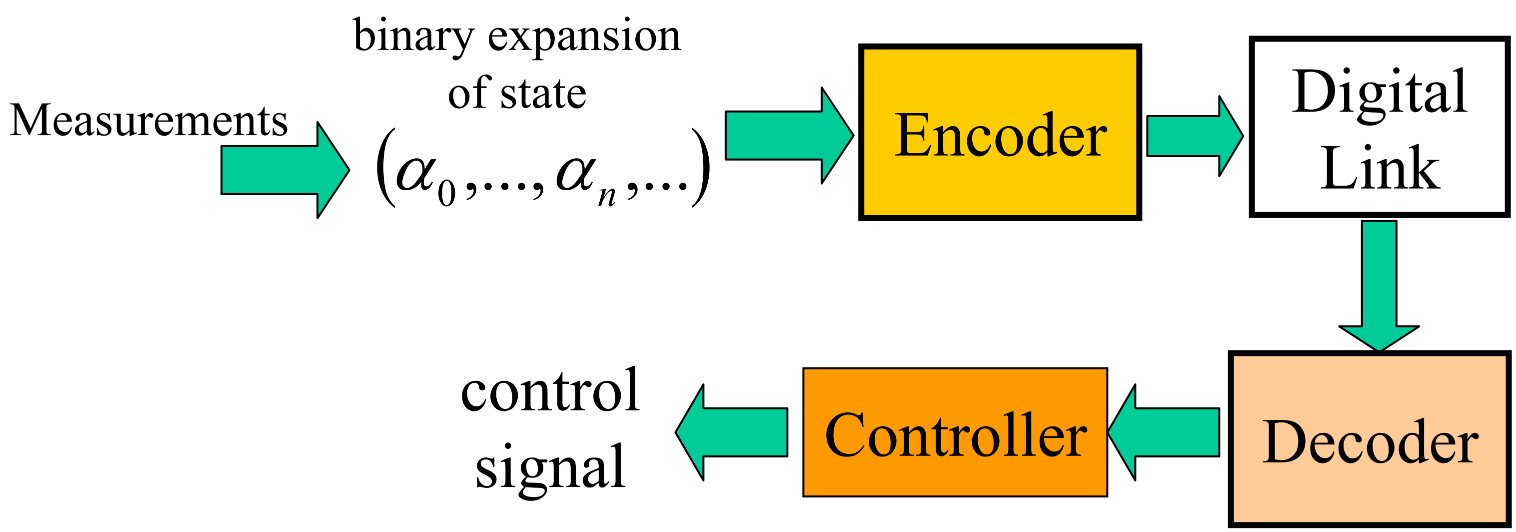
Almost sure internal stability

$$\Pr\left(\lim_{k\to\infty} \sup_{x_0\in[0,1]} |x_k| = 0\right) = 1$$

$C > R$	Sufficient condition
$C \geq R$	Necessary condition

- For this notion of stability, the quality of the link does not matter.
- Only the average of the random variables matter.

Sufficiency resorts to the following feedback scheme:



M-th Moment Stability

M-th moment internal stability

$$\lim_{k\to\infty} E\left[\sup_{x_0\in[0,1]} |x_k|^m\right] = 0$$

$C \geq R$ Necessary condition

- We assume:
  - a_k and r_k are I.I.D.
  - The following measure of system variation is finite:

$$\beta_m = \frac{1}{m}\log_2 E\left[2^{m\delta_a}\right]$$

- Similarly, we define the following channel reliability measure:

$$\alpha_m = \frac{1}{m}\log_2 E\left[2^{-m\delta_r}\right]$$

Main Result

Under such new assumptions, we can derive the following necessary and sufficient conditions:

$$C - \alpha_m \geq R + \beta_m$$

Necessary condition

$$C - \alpha_m > R + \beta_m$$

Sufficient condition

δ_a is normally distributed

In that situation:

$$C - \alpha_m \geq R + \frac{m\sigma^2}{2\ln 2}$$

Necessary condition

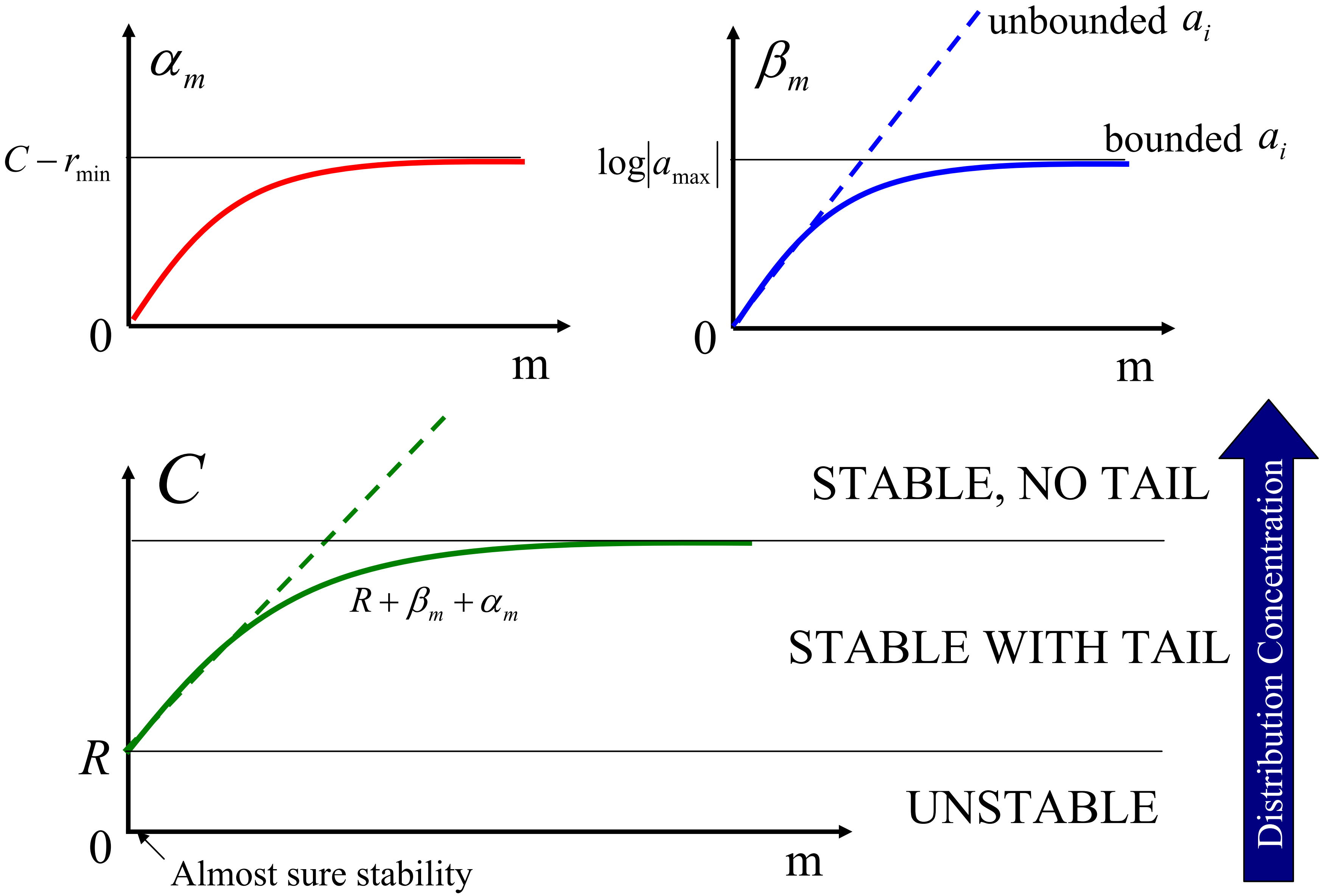
$$C - \alpha_m > R + \frac{m\sigma^2}{2\ln 2}$$

Sufficient condition

where

$$\sigma^2 = VAR\left(\log|a_k|\right)$$

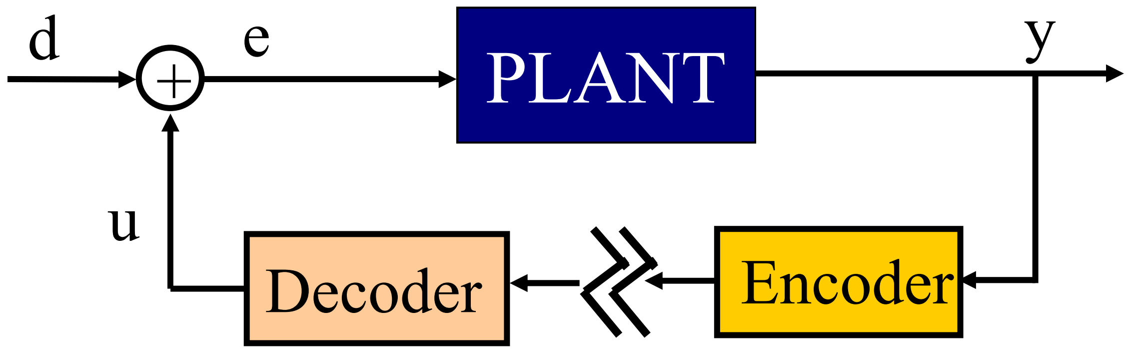
Properties of β_m and α_m



Implications of these results

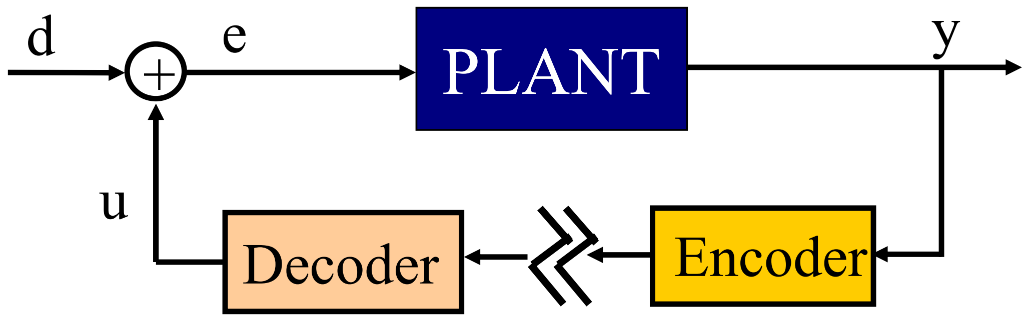
- Almost sure stability does not depend on the quality of the channel.
- Any almost-sure stable system is M-moment stable for m sufficiently small.
- For certain stochastic systems and channels, the states have fat tales.

Fundamental Limitations on Performance: Bode



$$\frac{1}{\pi}\int_0^{\pi}\left[\log|S|\right]_{-}d\theta+\frac{1}{\pi}\int_0^{\pi}\left[\log|S|\right]_{+}d\theta\geq\sum\max\left\{\log|\lambda_i|,0\right\}$$

Fundamental Limitations on Performance: Bode/Shannon



$$|S|^2 = \frac{S_e(e^{j\theta})}{S_d(e^{j\theta})}$$

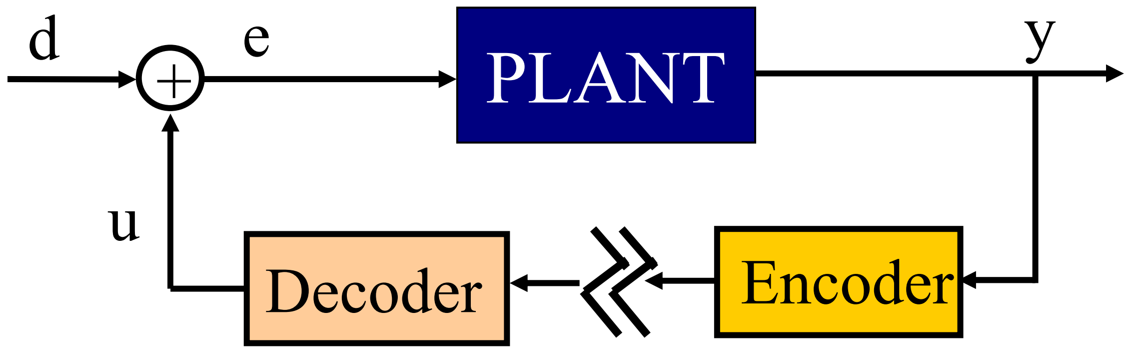
New integral formula for fundamental limitation in "sensitivity"

$$\frac{1}{\pi}\int_0^{\pi}\left[\log \gamma |S|\right]_- d\theta \geq -\lim \frac{1}{N} I\left(d^N; u^N\right)$$

Feedback Capacity

If d is Gaussian, then: $\gamma = 1$

Implications: Stability and Performance

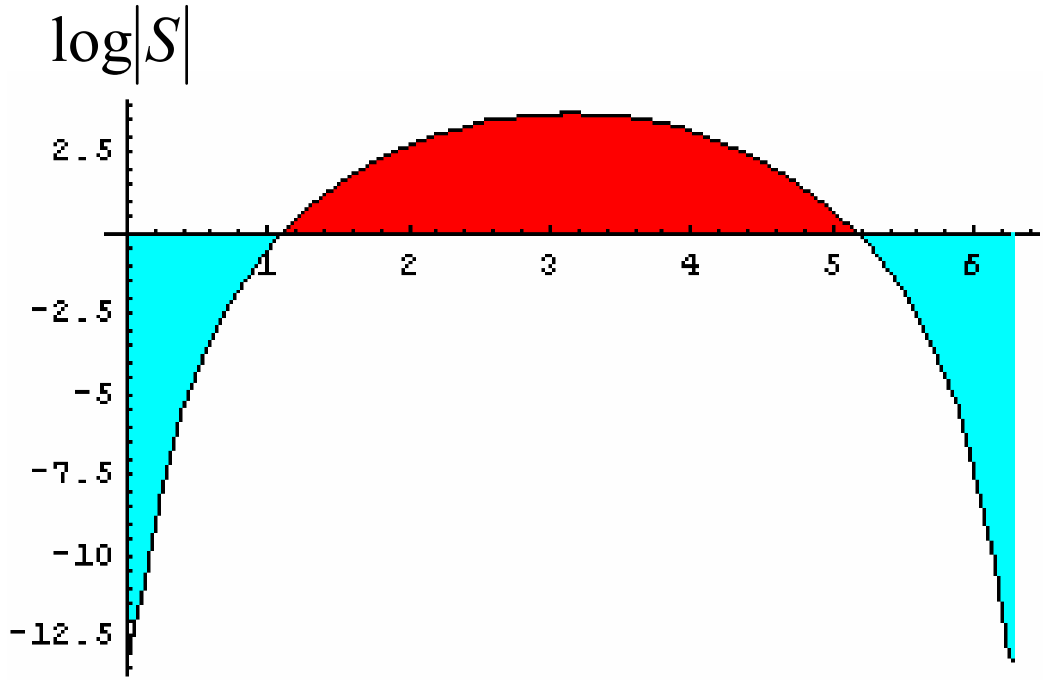


memory-less channel with
capacity C

$$\frac{1}{2\pi}\int_0^{\pi}\left[\log \gamma \frac{S_e\left(e^{j\theta}\right)}{S_d\left(e^{j\theta}\right)}\right]_{-} d\theta \geq -\lim \frac{1}{N} I\left(x^N; z^N\right) + \sum \max\{0, \log|\lambda_i|\}$$

$$\geq -C + \sum \max\{0, \log|\lambda_i|\}$$

Interpretation



(Bode)

$$\frac{1}{\pi}\int_0^{\pi}\left[\log|S|\right]_+ d\theta$$

$$\frac{1}{\pi}\int_0^{\pi}\left[\log|S|\right]_- d\theta$$

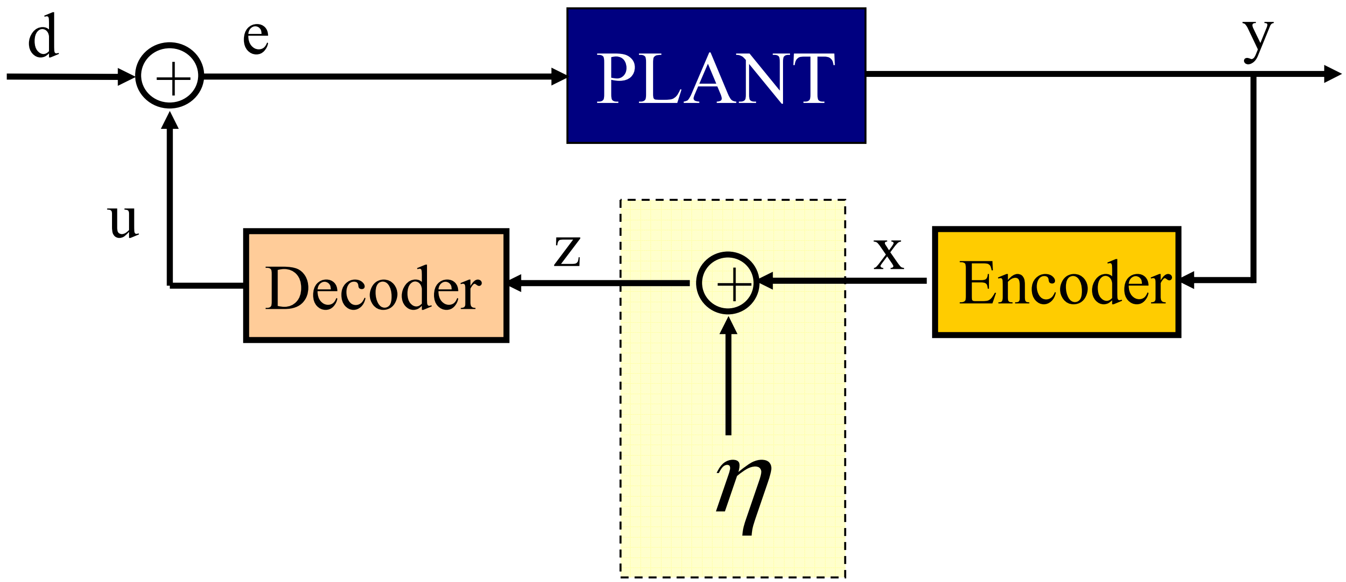
(Bode/Shannon)

$$\lim \frac{1}{N} I\left(x^N; z^N\right)$$

$$\frac{1}{\pi}\int_0^{\pi}\left[\log|S|\right]_- d\theta$$

both must be always active

Linear/Gaussian case



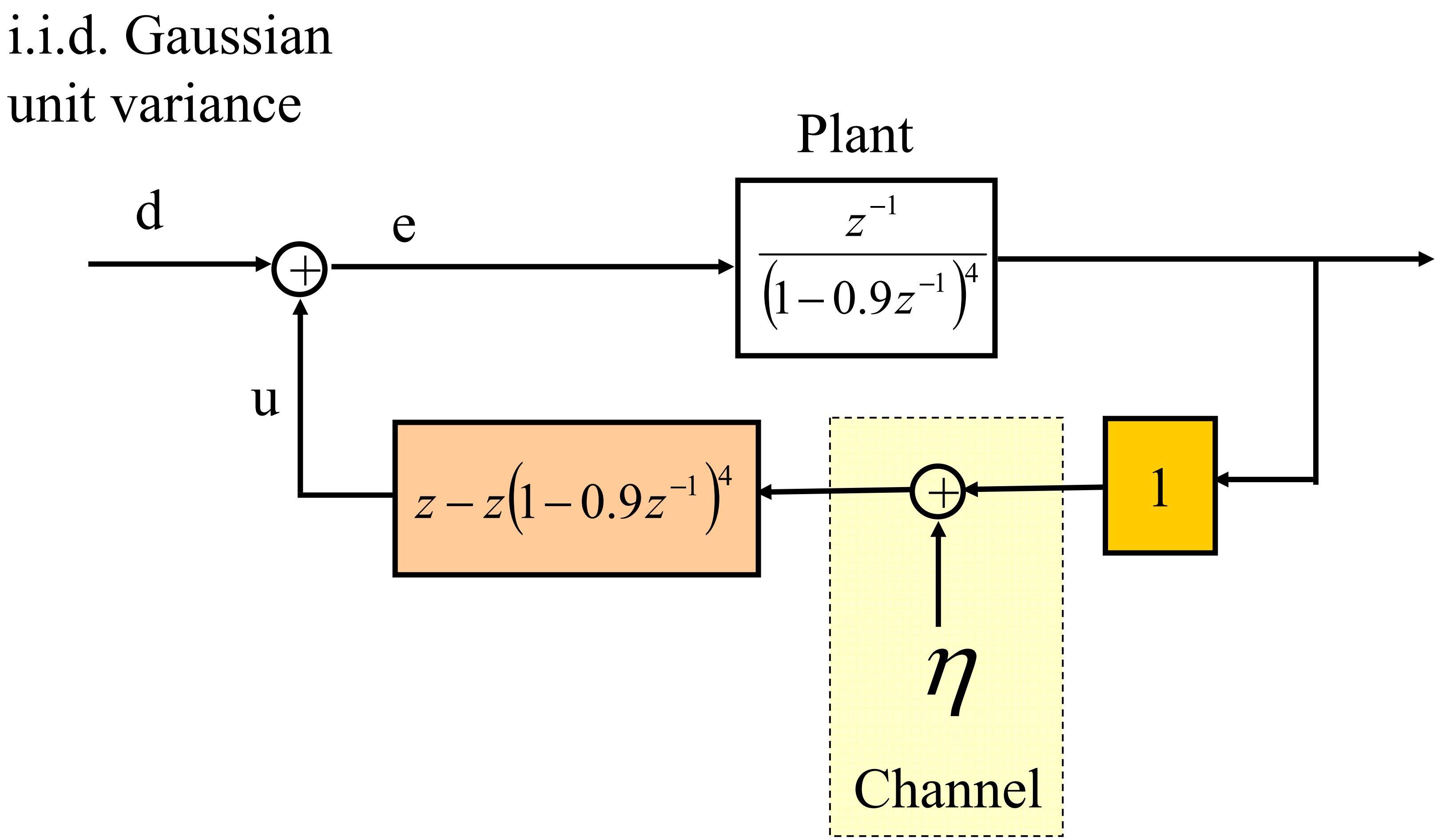
In the AWG case with linear encoder and decoder, the following holds!

$$\lim \frac{1}{N} I\left(u^N, d^N\right) = \lim \frac{1}{N} I\left(z^N, x^N\right) - \sum \log\left|O.L.UnstablePoles\right|$$

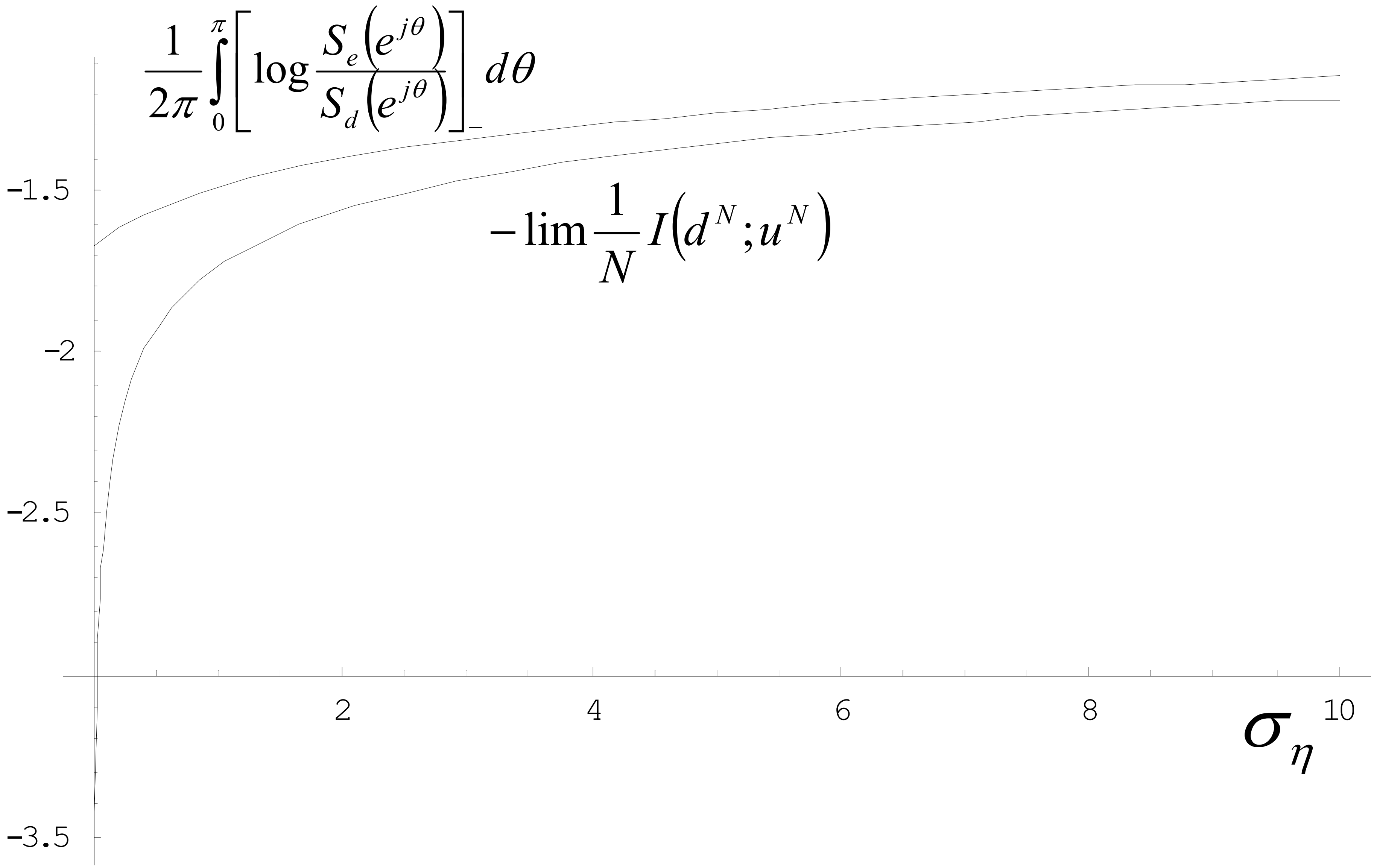
Leads to: $$\frac{1}{2\pi}\int_0^{\pi}\left[\log \frac{S_e\left(e^{j\theta}\right)}{S_d\left(e^{j\theta}\right)}\right]_- d\theta \geq -C + \sum \log\left|O.L.UnstablePoles\right|$$

Bound is Tight!

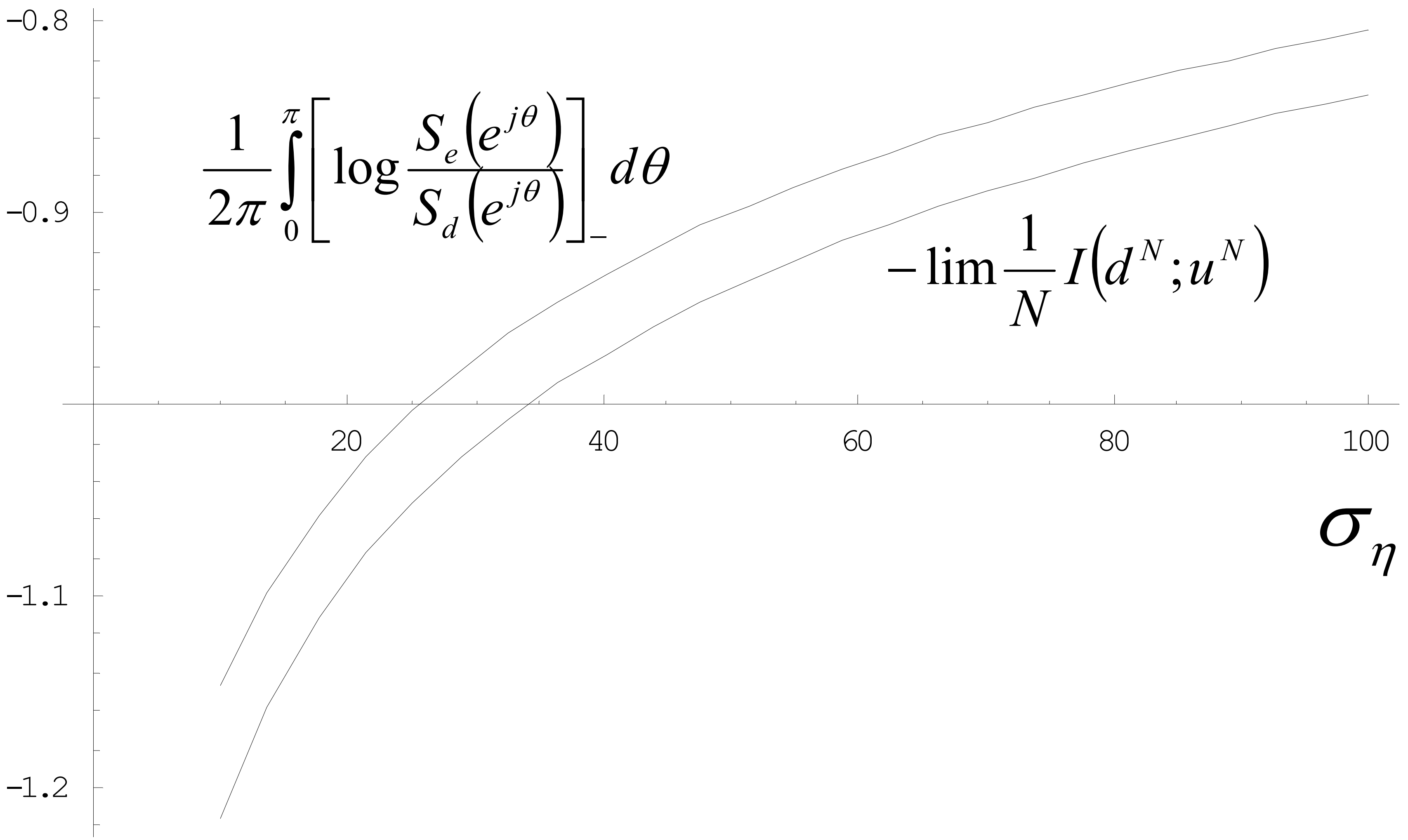
Example:



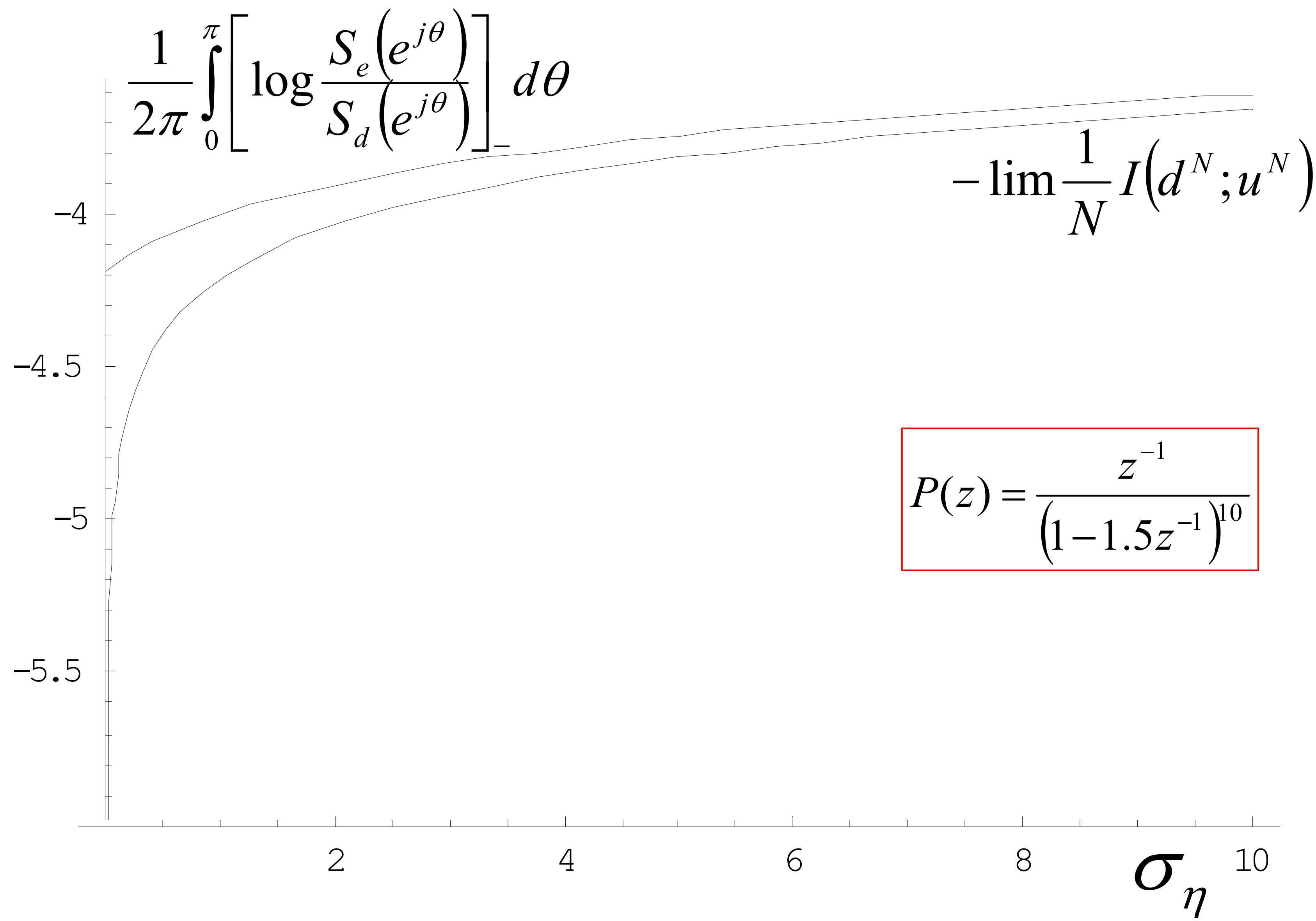
Bound is Tight: Numerical results



Bound is Tight: Numerical results

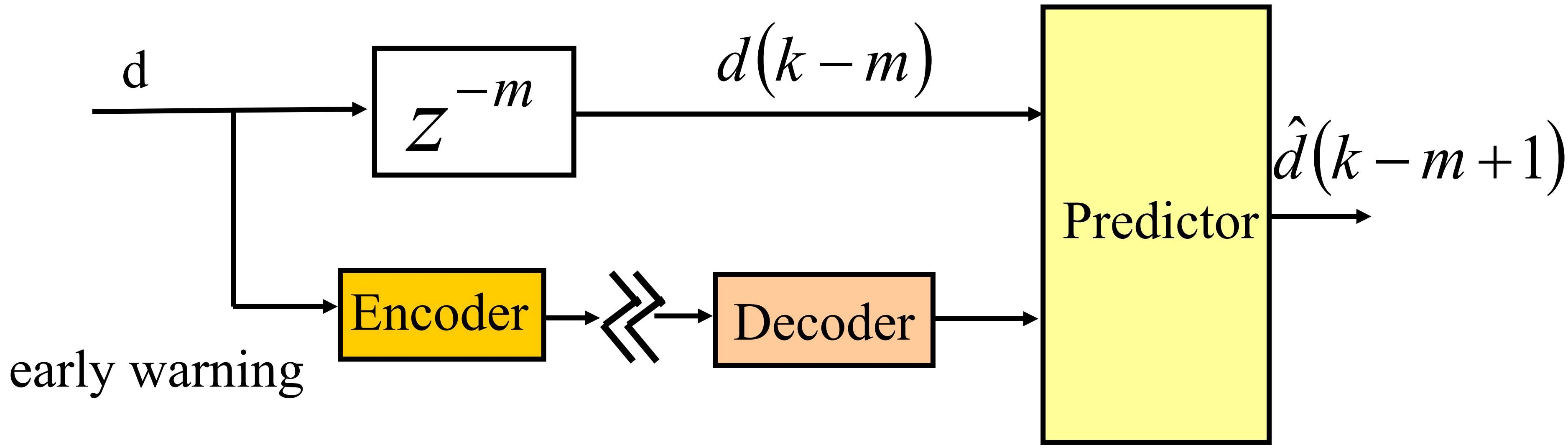


Bound is Tight: Another example



Forward loop: One step prediction with side information

(Problem motivated and developed in collaboration with John Doyle)



$$\frac{1}{2\pi}\int_0^{\pi}\log\frac{S_e\left(e^{j\theta}\right)}{S_d\left(e^{j\theta}\right)}d\theta \geq -C$$

Achievable for the AW Gaussian channel.

Conclusions

- There is a minimal critical capacity that is necessary and sufficient for stabilization in the almost sure sense. The quality of the link and the variability of the environment do not impact the result.

- M-th moment stability requires more information. If r_k and a_k are I.I.d. then the critical capacity depends on the link quality and the measure of environmental variation.

- For certain distributions, states can have heavy tails.

- Performance -Capacity Tradeoff: A bode-like integral.