Myths, Misconceptions and Misuses of Nonlinearity

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The Issues

Observed Phenomena . . . . Mathematical Models

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- Phenomena: Data, system trajectories, time traces
  Models: Differential equations

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  - Complexity of phenomena (scale, patterns, etc.)

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- Phenomena: Data, system trajectories, time traces
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- Concern: Complexity
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    somewhat subjective
  - Complexity of models (?????)

    Linear vs. Nonlinear

    Low vs. High order

    . . . .

Outline

- A story of boundary layer turbulence

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  - Effective analysis using tools from linear systems theory

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  - Model order
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- Are linear and nonlinear models equivalent?
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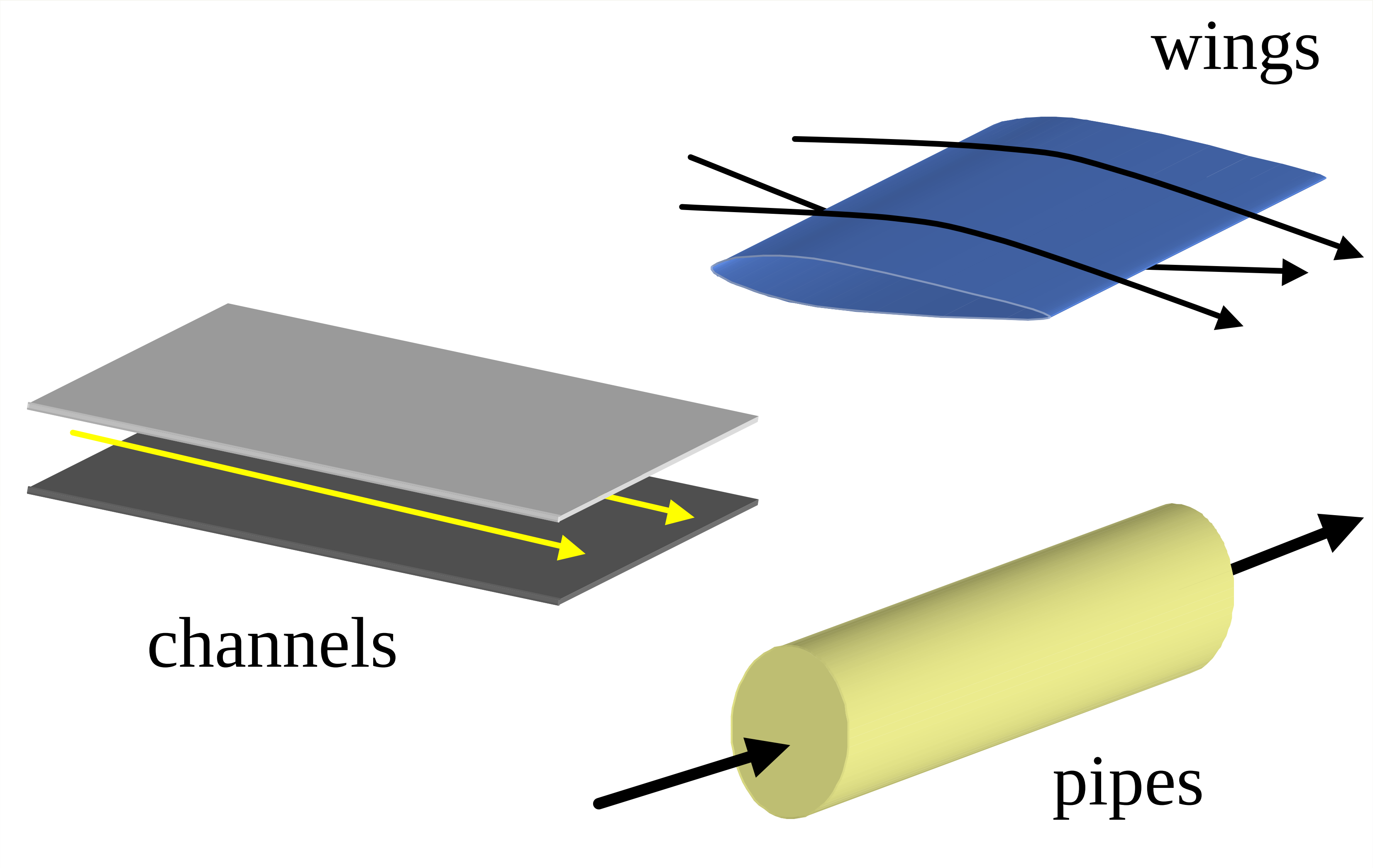
  - Special structure

- The use of “nonlinearity” in current scientific culture
  - a runaway concept

The phenomenon of turbulence

Technologically important flows: *Flows past streamlined bodies*

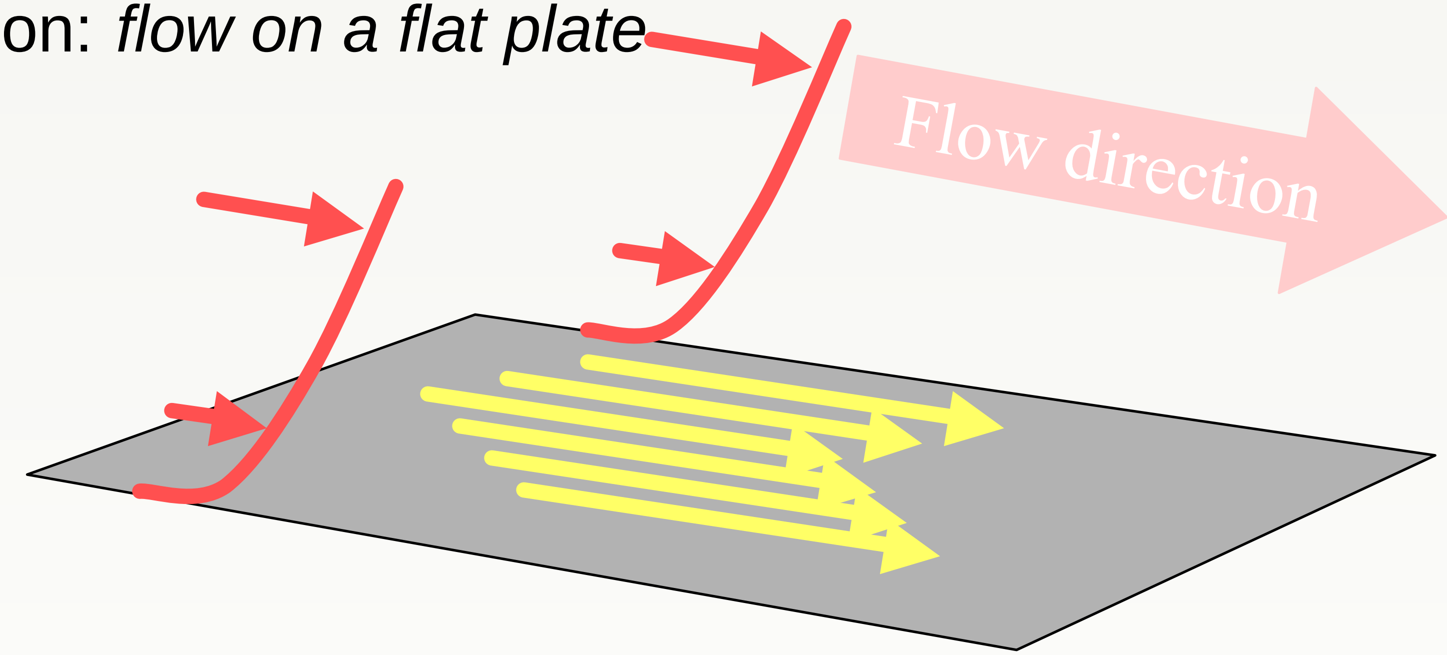


Wall-bounded shear flows *Friction with the walls drives the flows*

The phenomenon of turbulence (Cont.)

Boundary layers form in flow past any surface

Idealization: *flow on a flat plate*



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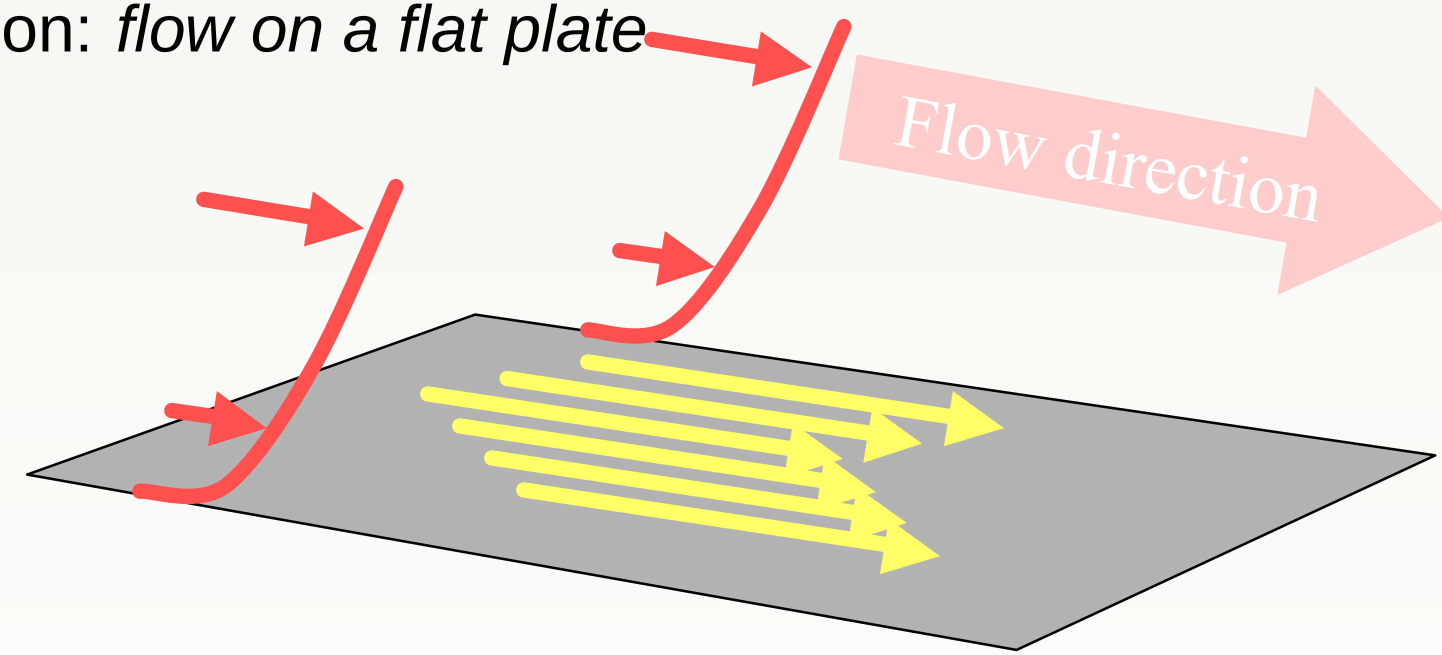
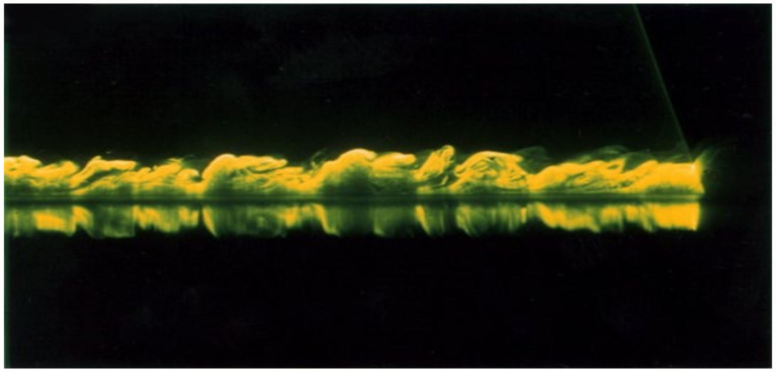
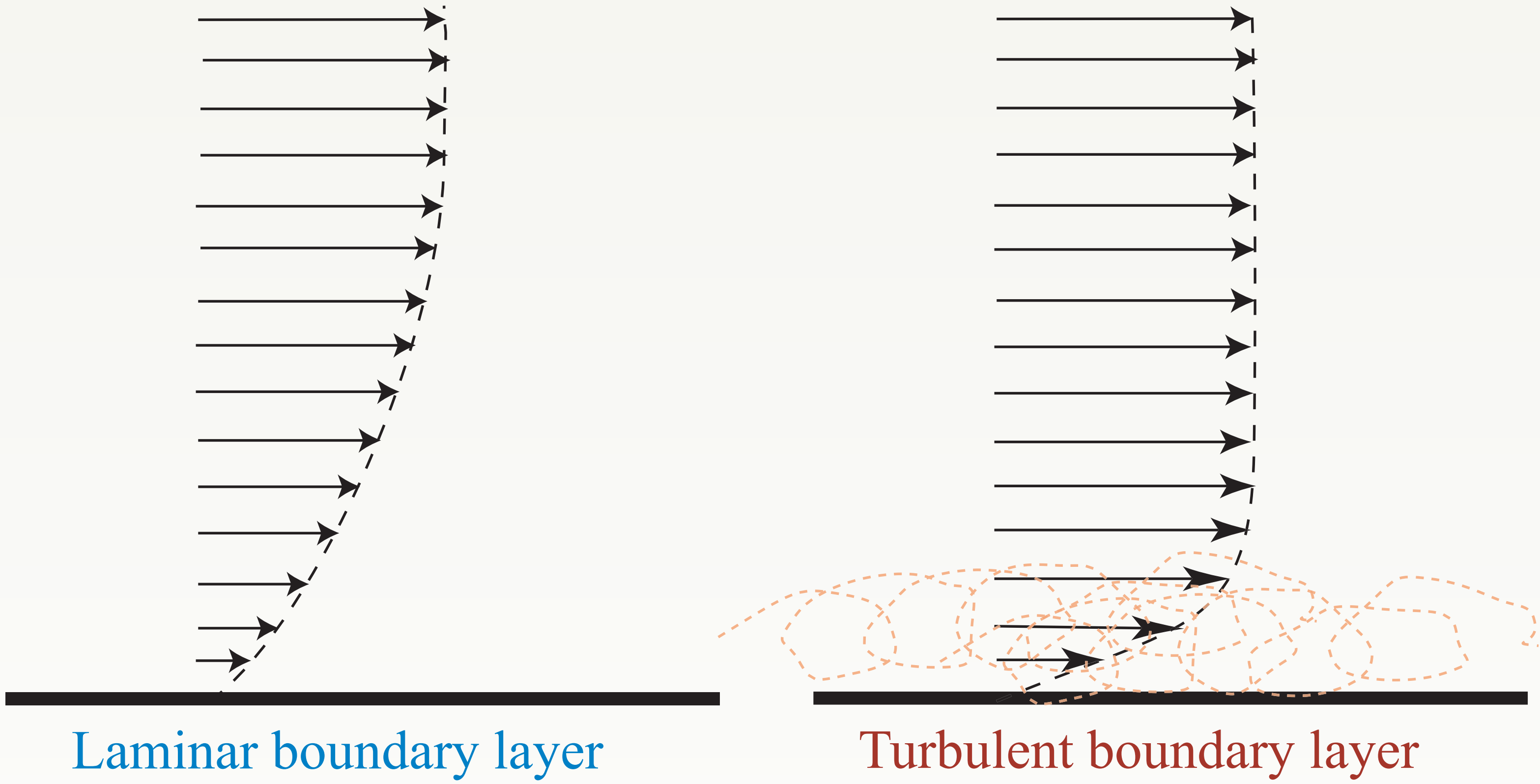


Viewed sideways

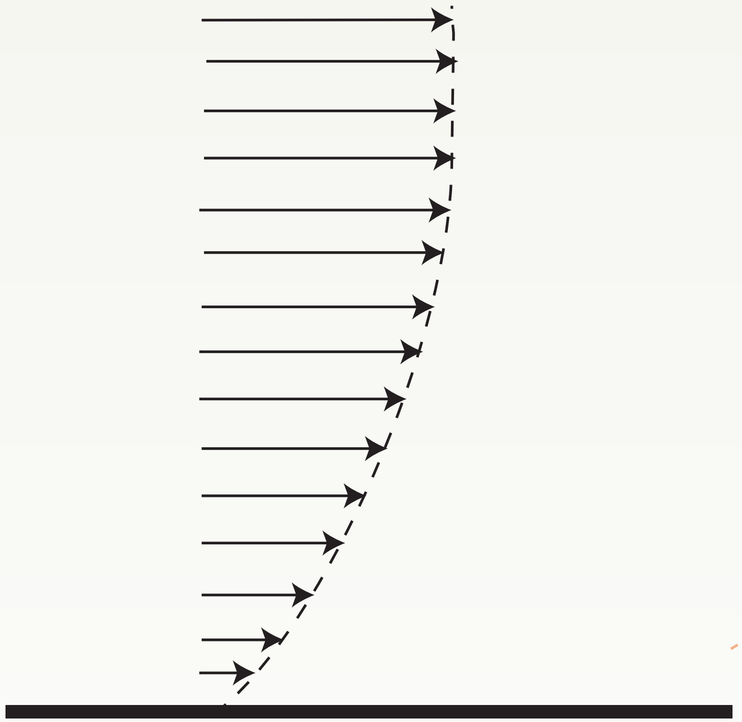


Boundary layer turbulence and skin-friction drag

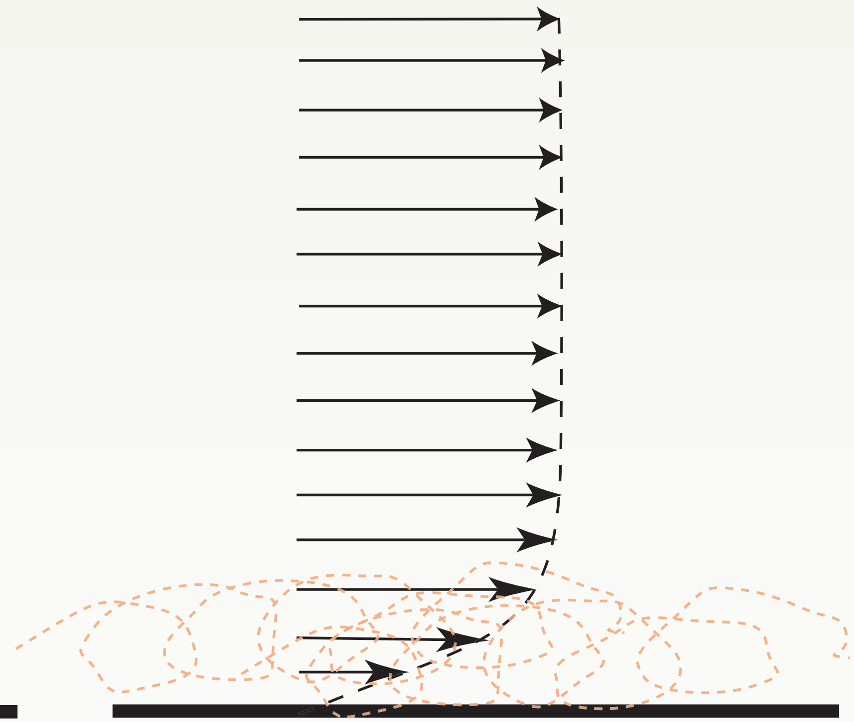


A laminar BL causes less drag than a turbulent BL (for same free-stream velocity)

Boundary layer turbulence and skin-friction drag



Laminar boundary layer



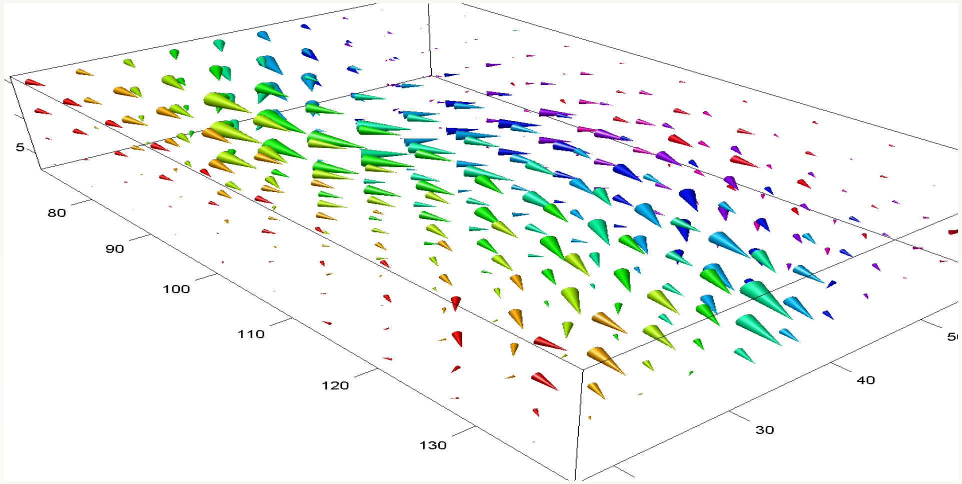
Turbulent boundary layer

A laminar BL causes less drag than a turbulent BL (for same free-stream velocity)

This *skin-friction drag* is 40-50% of total drag on typical airliner

The "Dynamical Systems" view

The flow field at time time $=: \quad \mathbf{\Psi}(t)$



$\mathbf{\Psi}(t)$ is the vector field of flow velocities

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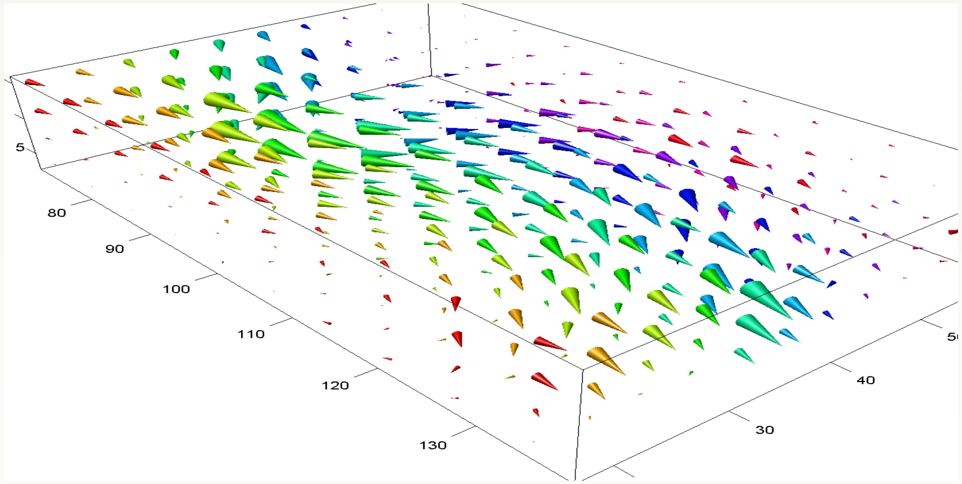

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Fluid dynamics (e.g. *Navier-Stokes equations*) can be written as:

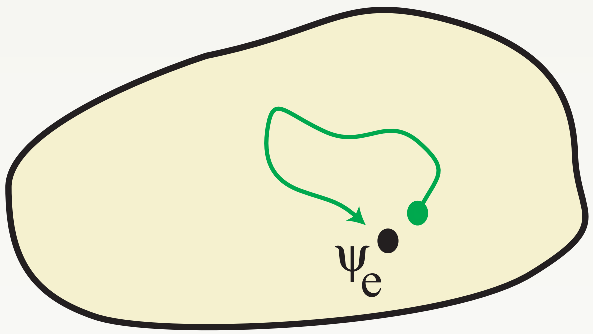
$$\partial_t \mathbf{\Psi}(t) \qquad = \quad \mathcal{F}\Big(\mathbf{\Psi}(t), R\Big)$$

$\uparrow$ change in time

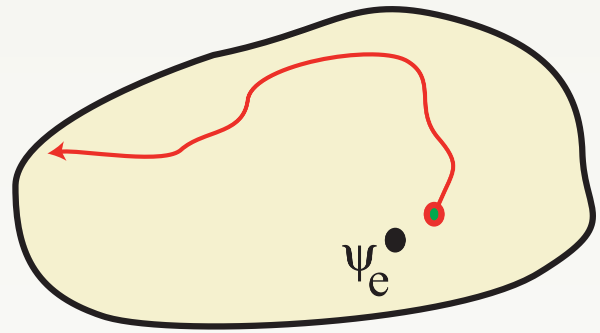
$\uparrow$ function of current state

Does the system stay near an equilibrium?

If starting “near” equilibrium, does system come back to it??



stable equilibrium



unstable equilibrium

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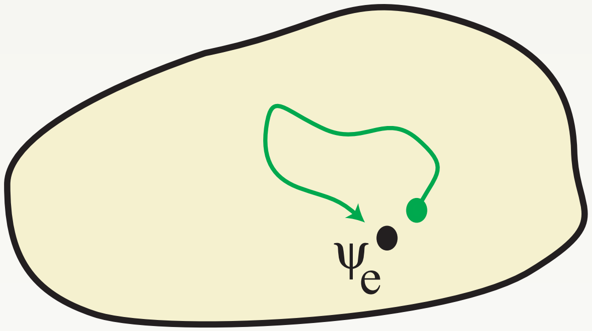

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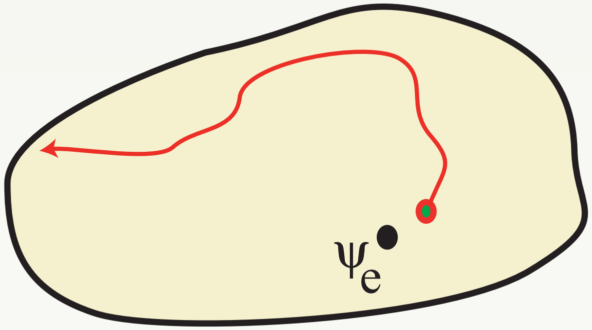

unstable equilibrium

An unstable equilibrium is not really an "equilibrium"

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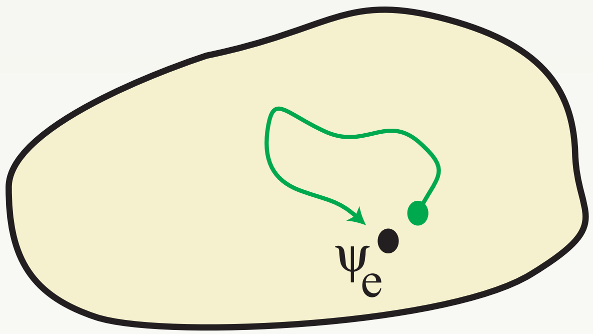

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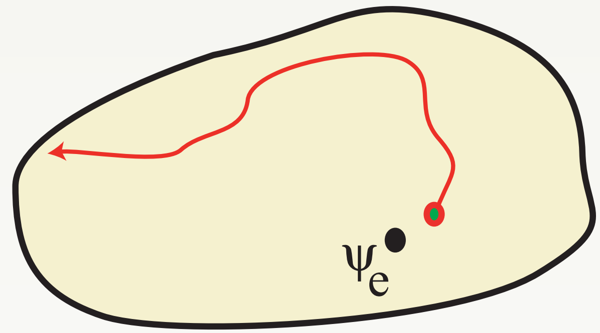

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How to check stability?

Common method: Linearization

$$\partial_t \mathbf{\Psi}(t) = \mathcal{F}\Big(\mathbf{\Psi}(t)\Big) = \underset{\substack{\uparrow \\ \text{Linear part}}}{\mathcal{A}}\Big(\mathbf{\Psi}(t)\Big) + \underset{\substack{\uparrow \\ \text{The rest}}}{\mathcal{N}}\Big(\mathbf{\Psi}(t)\Big)$$

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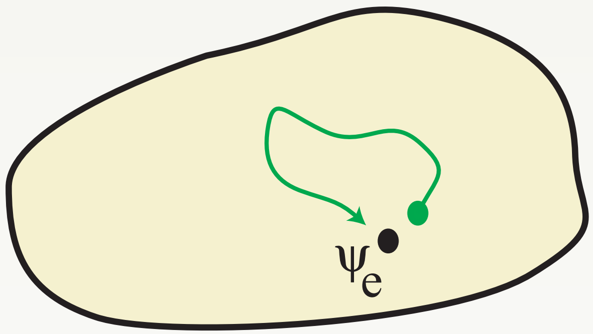

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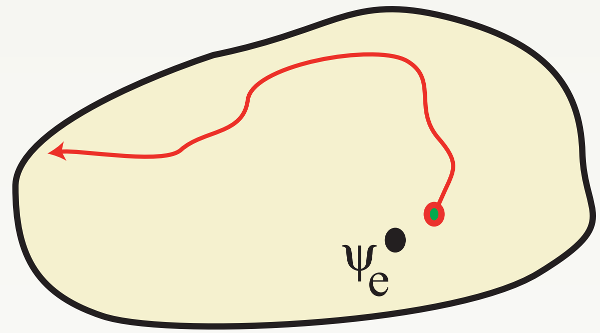

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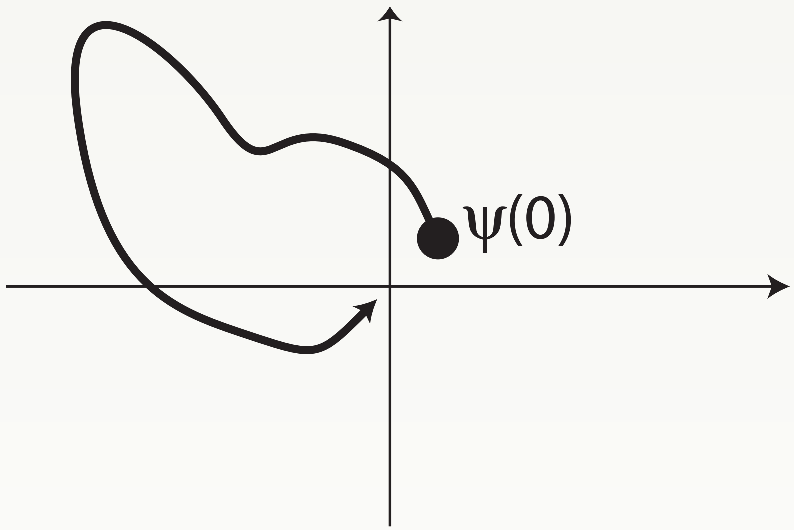
Stability (instability) of $\mathcal{A}$ $\quad\Leftrightarrow\quad$ Local stability (instability) of $\mathcal{F}$.

Can be studied using eigenvalue/eigenfunction analysis of $\mathcal{A}$

Uncertainty in a Dynamical System

Lyapunov Stability deals with **uncertainty** in initial conditions



If $\mathbf{\Psi}(0)$ is known to be *precisely* $\mathbf{\Psi}_e$, then $\mathbf{\Psi}(t) = \mathbf{\Psi}_e,\ t \geq 0$

We introduce the concept of Lyapunov stability because we can never be *infinitely certain* about the initial condition

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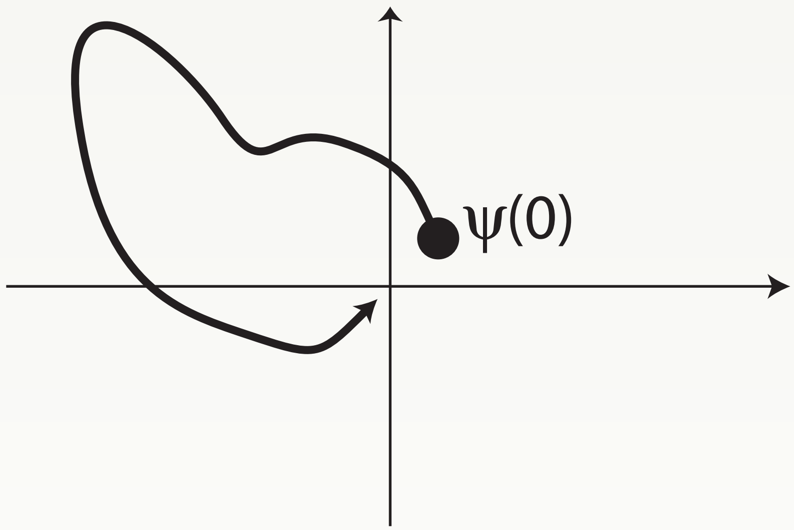

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Shortcomings of Lyapunov stability

- Perturbs only initial conditions
- Cares only about asymptotic behavior

Uncertainty in a Dynamical System (cont.)

Lyapunov stability

$$\dot{\psi} = f(\psi)$$

uncertain initial conditions

investigate $\lim_{t \to \infty} \psi(t)$

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Linearized version:

transient growth

umodelled dynamics

$\dot{\psi} = (A + \Delta)\psi$

Psuedo-spectrum

eigenvalue stability

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combinations

$$\dot{\psi}(t) = (A + B\Delta C)\psi(t) + (F + G\Delta H)d(t)$$

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input-output analysis

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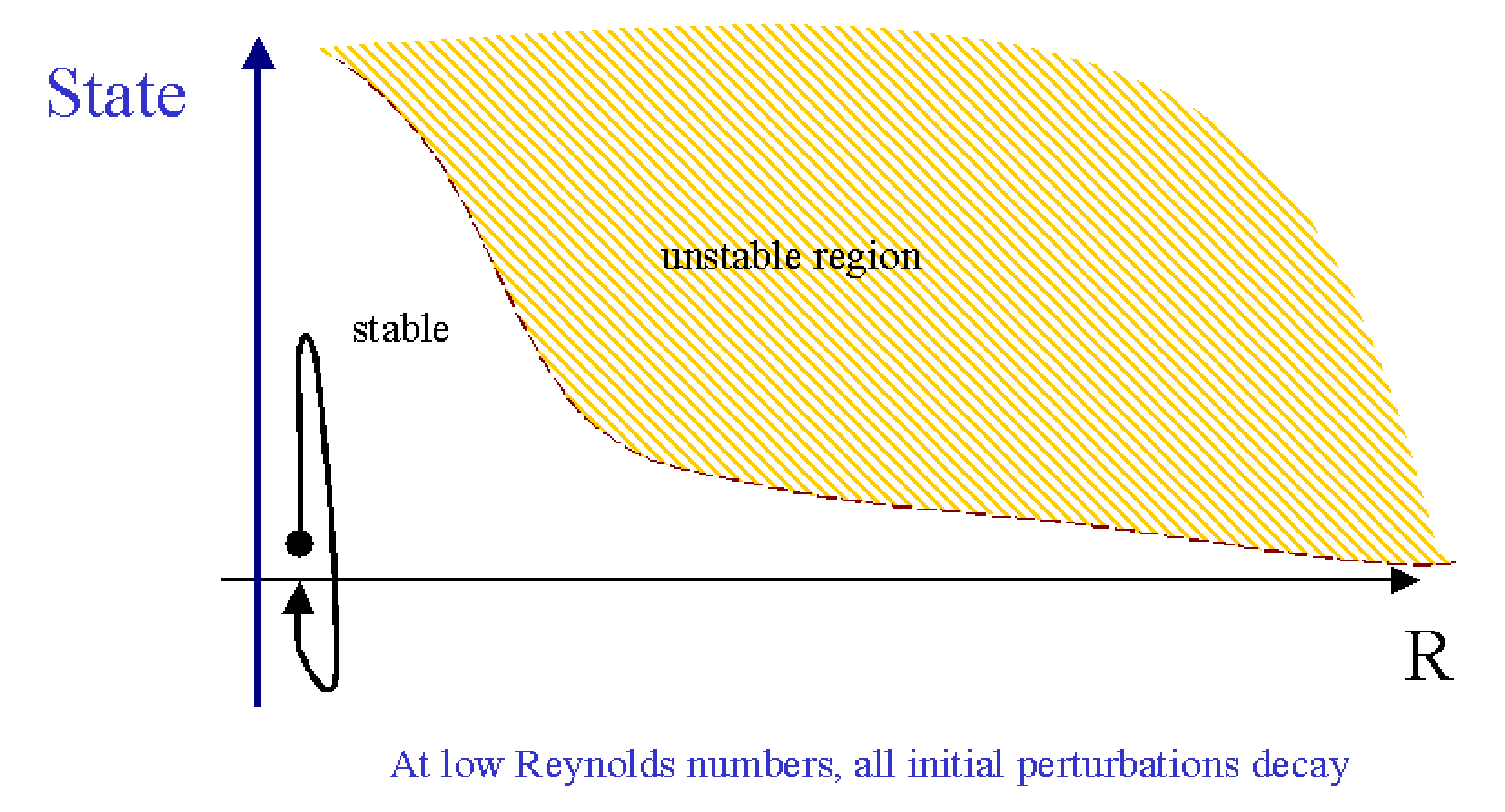
The nature of turbulence

Fluid dynamics are described by deterministic equations

Why does fluid flow “look random” at high Reynolds numbers??

The nature of turbulence

Common view of turbulence



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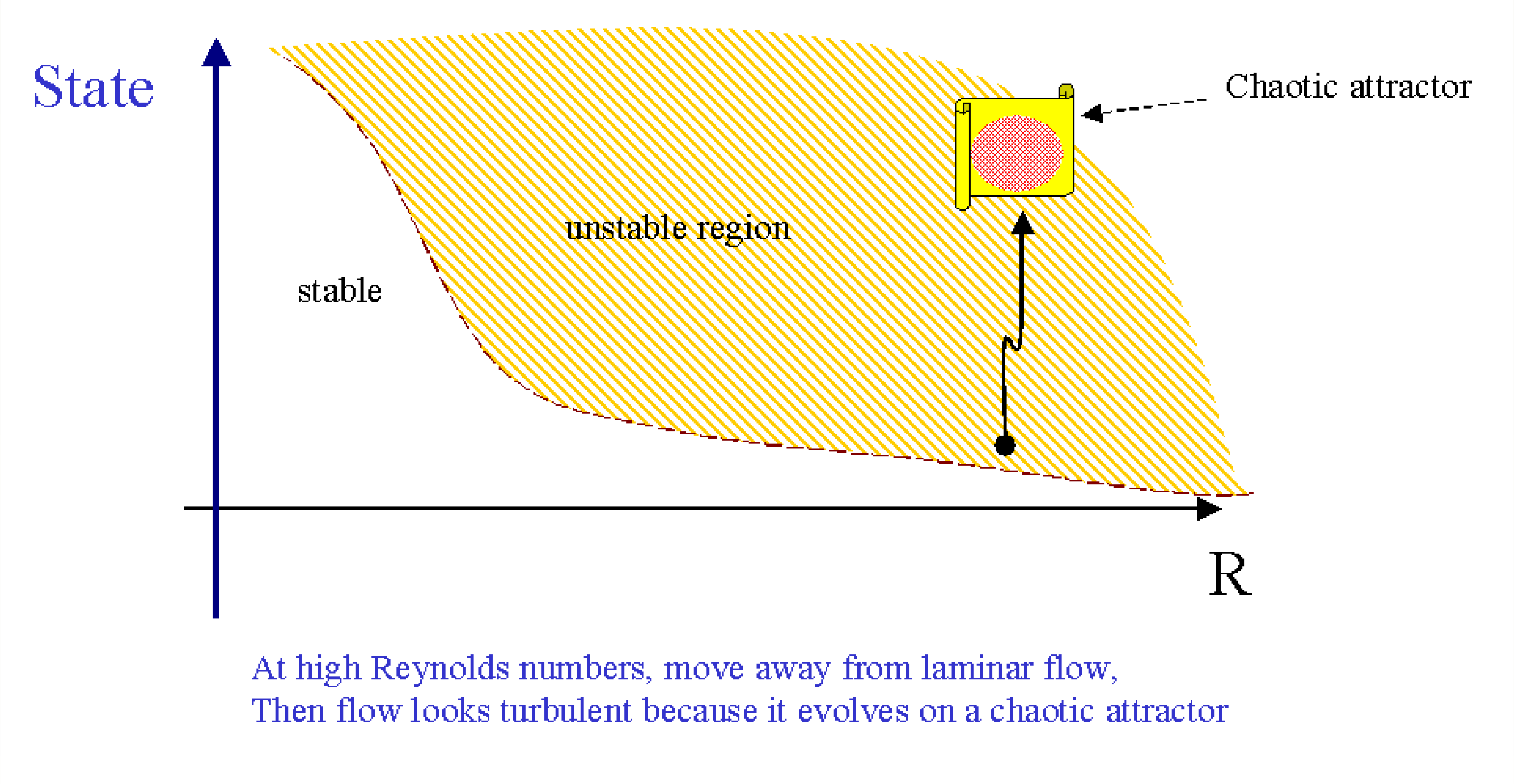

Intuitive reasoning

Complex, statistical looking behavior $\longleftrightarrow$ System with chaotic dynamics

The nature of turbulence (cont.)

Alternative view

Noise
Surface roughness
Thermal forces
Free stream disturbances

Dynamics of the Navier-Stokes Equations

Fluctuating flow field

(looks statistical)

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Surface roughness
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→ **Dynamics of the Navier-Stokes Equations** →

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Qualitatively similar to

Noise
Surface roughness
Thermal forces
Free stream disturbances

→ **Dynamics of the *Linearized* Navier-Stokes Equations** →

Fluctuating flow field

(looks statistical)

So what now?

- A story of boundary layer turbulence
  - Effective analysis using tools from linear systems theory
  - “but I thought turbulence was a nonlinear phenomenon!!??”
- Are linear and nonlinear models equivalent?
  - Linearization techniques
  - Model order
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Linearization techniques

The Carleman Linearization

Example: $\frac{d}{dt}x \ = \ x^2$

Define: $x_1 := x, \ \ x_2 := x^2, \ \ , \cdots , \ x_n := x^n, \ \ \cdots$

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Then

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 2 & 0 & \\ 0 & 0 & 0 & 0 & 3 & \\ & & \vdots & & & \ddots \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{bmatrix}, \qquad \text{i.e.} \qquad \dot{\mathcal{X}} = \mathcal{A}\mathcal{X}$$

The original system is imbedded in this linear system

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General procedure: Given $\dot{x} = F(x, u)$

Define $\mathcal{X}$ with components $x^i u^j$, then

$$\dot{\mathcal{X}} = \mathcal{A}\mathcal{X} + (\dot{u})\, \mathcal{B}\mathcal{X}$$

Linearization techniques (cont.)

- **The Lie-Koopman Linearization**

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- **State transformations** and **feedback linearization**

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- $\cdots\cdots\cdots$

Linearization techniques often do not make a problem more tractable

e.g. computing $e^{t\mathcal{A}}$ can be arbitrarily complex!!

Linear Phenomena

Q: What phenomena can a linear dynamical system explain?

$$\frac{d}{dt}\Psi(t) \; = \; \mathcal{A} \, \Psi(t)$$

A: Any phenomena explainable using a nonlinear dynamical system

Linear Phenomena

Q: What phenomena can a linear dynamical system explain?

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A: Any phenomena explainable using a nonlinear dynamical system

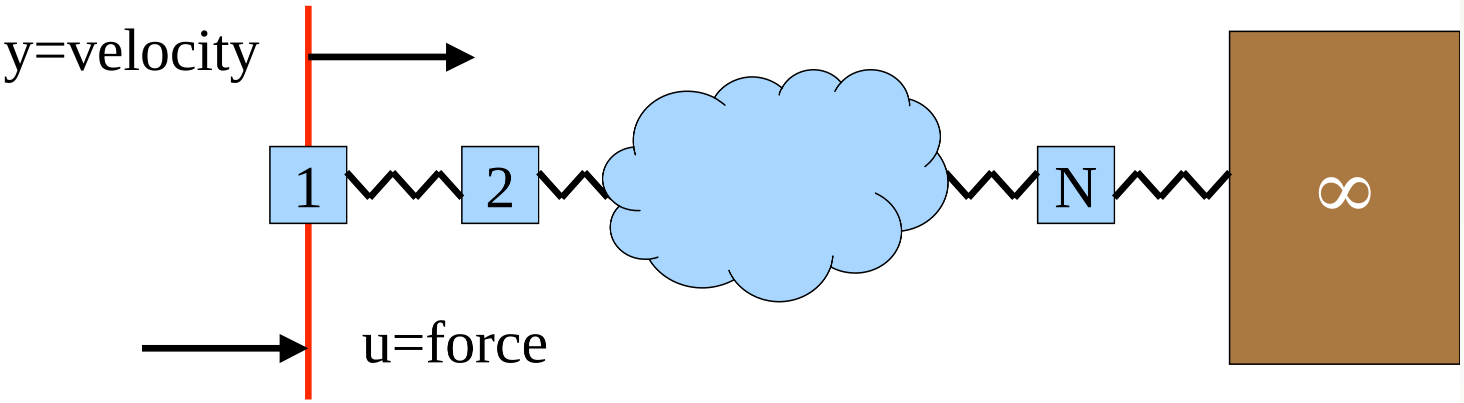
The term “nonlinear phenomenon” has no mathematical meaning!

“Chaos” in Linear Systems

Homogeneous N masses, N+1 springs

Mass=1

Spring K=1



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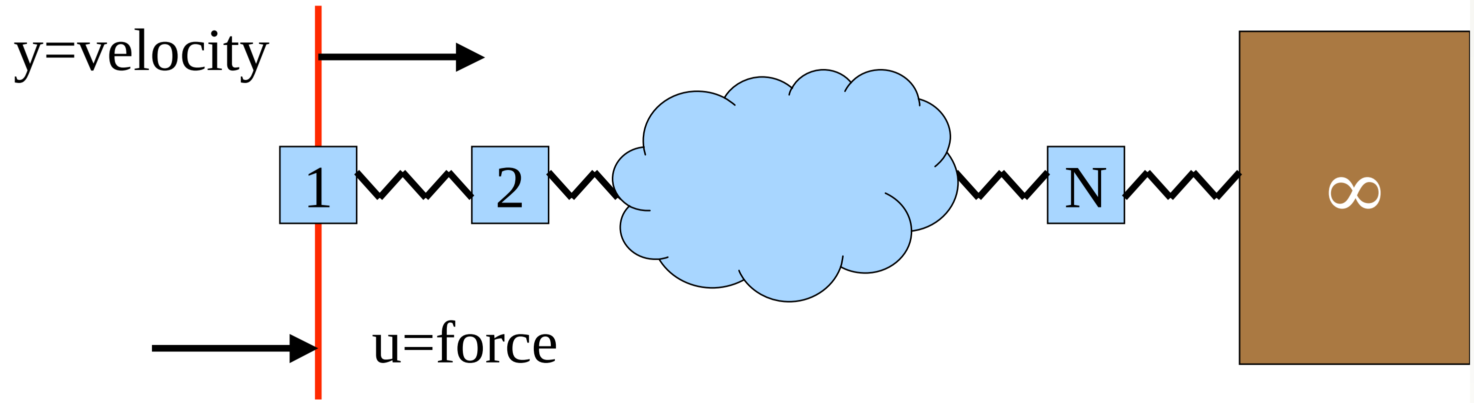

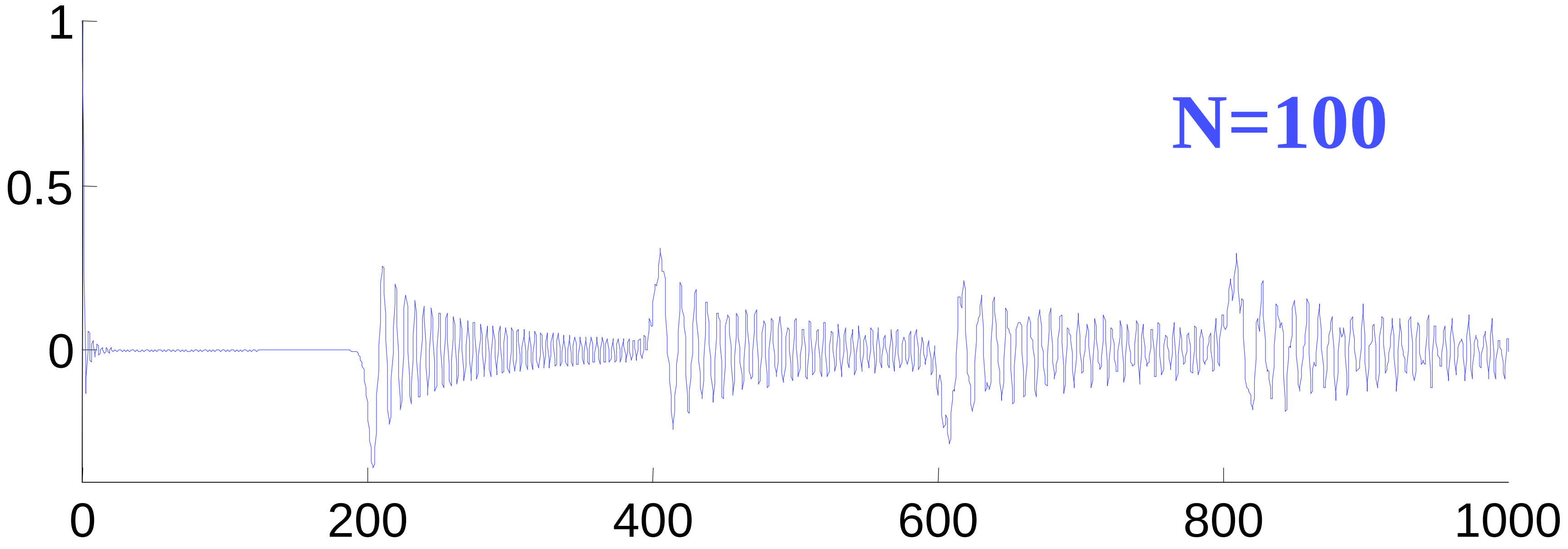

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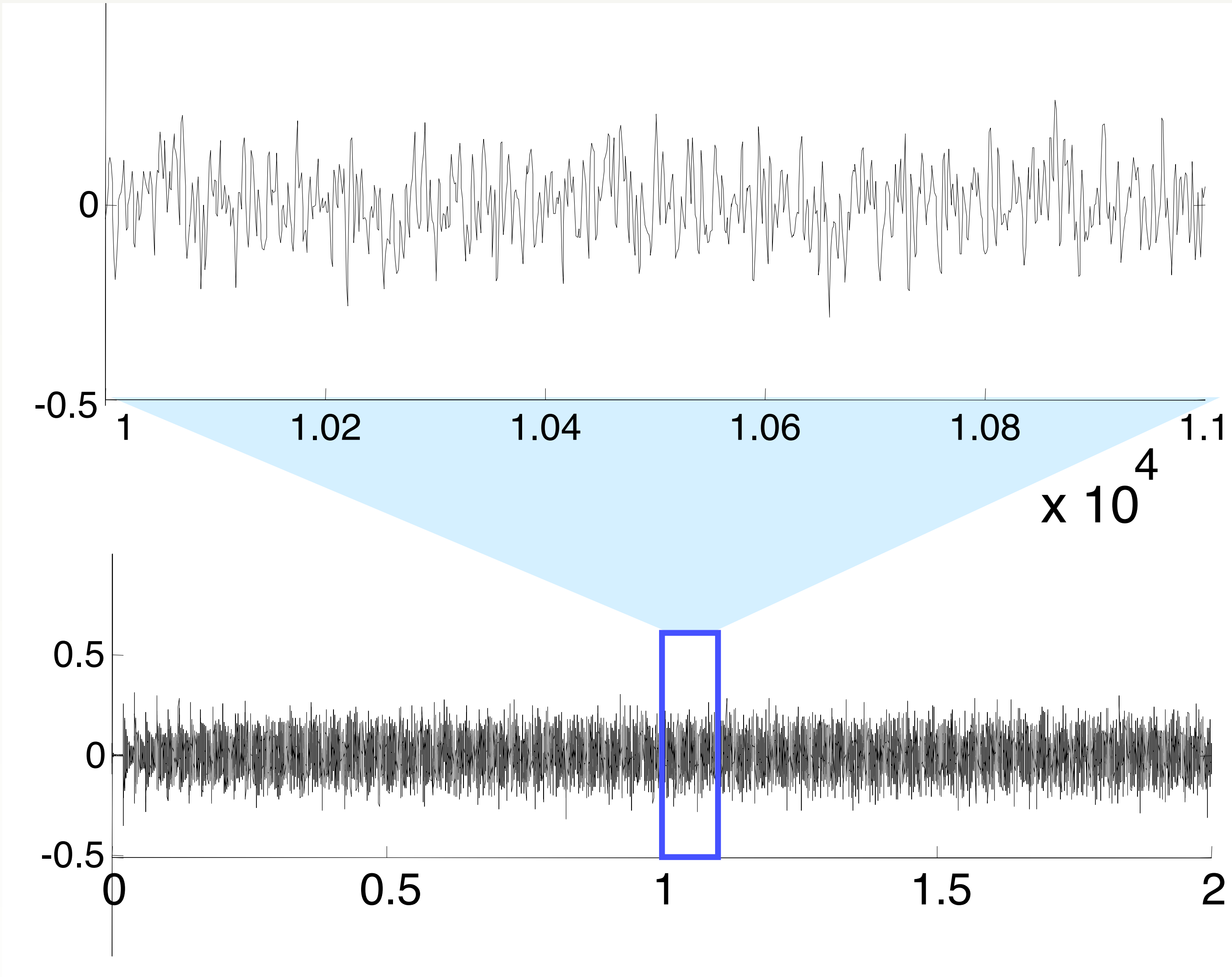

has continuous spectrum (PSD) over long, but finite times

“Chaos” in Linear Systems (cont.)

- With similar schemes,
  can generate stochastic processes with any prescribed PSD
- Clearly not the most efficient method to generate such processes!

Moral: *Sometimes a low dimensional nonlinear model is easier to handle than a high dimensional linear one*

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Moral: *Sometimes a low dimensional nonlinear model is easier to handle than a high dimensional linear one*

Another example: *Optimal filtering equations*

Phrases to avoid

A well defined mathematical object: **nonlinear mapping**

mapping between two vector spaces that does not preserve their linear structure

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- Even the term “nonlinear system” is a little ambiguous

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To make sure you are on firm ground:

use the term nonlinear only in its original mathematical meaning

The End