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On the generation of Whitney forms

Just a chain map, from singular to simplicial chains

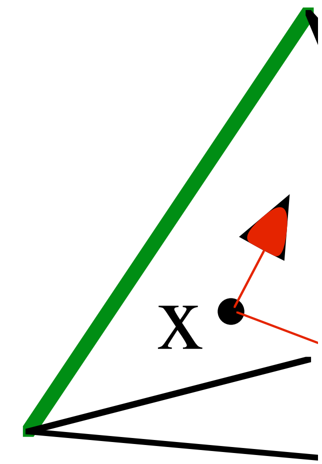
$$\square\,\partial_t A + \mathrm{rot}(\square\,\mathrm{rot}\,A) = J^s$$

$$(B = \mathrm{rot}\ \ldots$$
$$E = -\,\partial_t A\ldots$$

$$\ldots_t \square\square\ \square \cdot \square' + \square\square\,\mathrm{rot}\,\square \cdot \mathrm{rot}\,\square' = \square J^s \cdot \square'$$

$$\square\ \square'\ \square\ \ldots$$

- A : Finite-dimensional space of curl-conformal elements
- Edge circulations, physically meaningful, should be degrees of freedom



W(x) = a …

Then:

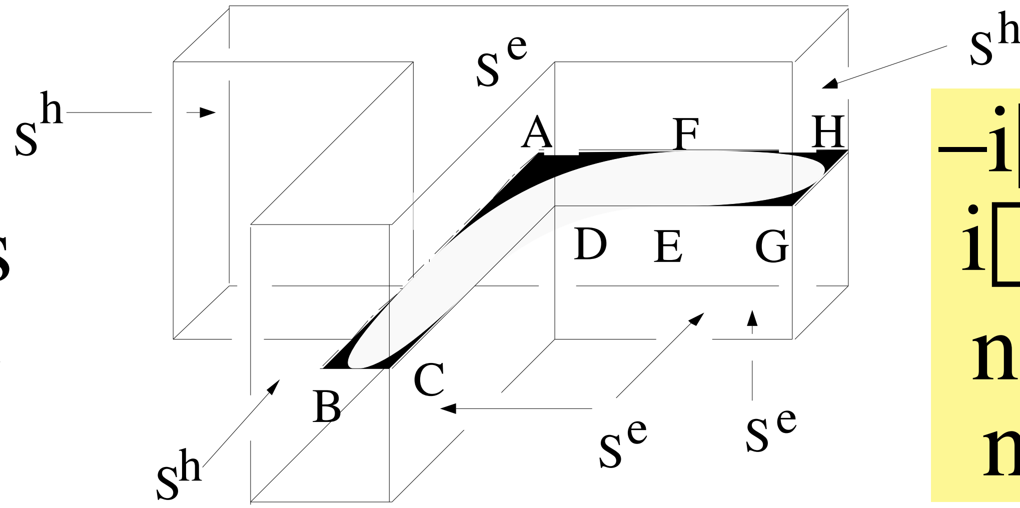
$$\square\,\partial_t \mathbf{a} + \mathbf{M}\,\mathbf{a} = \mathbf{j}^s$$

Now:

$$\square\,\partial_t \mathbf{a} + R^t \square R\, \mathbf{a}\ldots$$

he infamous "spurious modes", ca. 1

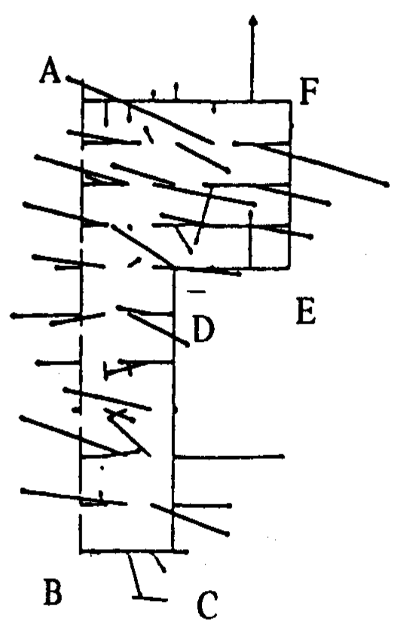
Compute
resonant modes
n a waveguide
T-junction

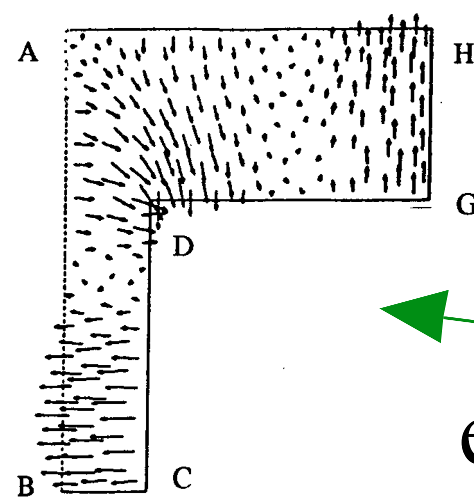


–i□□E + r
i□□H + r
n □ H = (
n □ E = (

View of field E in shaded plane section:

Jsing standard
ode-based
ector-valued
lements





edge e

$$\mathrm{rot}\left(\frac{1}{\square}\mathrm{rot}\ \square\right) = \square^2 \square\square \quad \square \quad \mathrm{div}(\square\square) =$$

only weakly enfor

Find □ *in* $\mathcal{E}$ (whose definition includes $n \square \square = 0$ on S^e) *s*

$$\square\mathrm{rot}\ \square \cdot \mathrm{rot}\ \square' = \square^2 \square\square\square \cdot \square' \quad \square \quad \square' \quad \square$$

Set $\square' = \mathrm{grad}\ \square'$ □ $\square\square\square \cdot \mathrm{grad}\ \square' = 0$ □ $\square'$ □

the weak form of $\mathrm{div}(\square\square) = 0$, so require grad

with □ large enough. Not the case if □ span

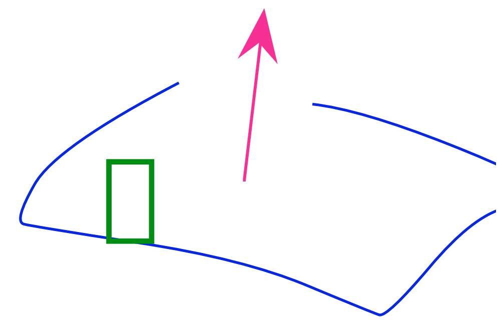
nodal vectorial elements. Whereas if $E = W^1$,

□

$$W^0 \xrightarrow{\mathrm{grad}} W^1 \xrightarrow{\mathrm{rot}} W^2 \xrightarrow{\mathrm{div}}$$

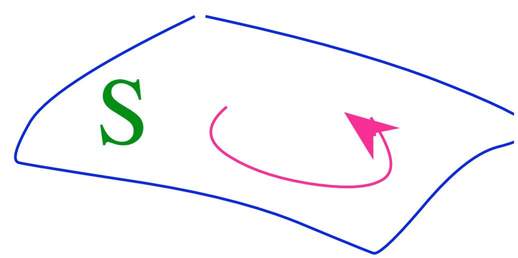
Maxwell, in terms of DF's:

$$- \partial_t \square_{\square} d + \square_{\partial \square} h = \square_{\square} j \quad \square$$



$$b = \square h \qquad d = \square e$$

$$\partial_t \square_{S} b + \square_{\partial S} e = 0 \quad \square$$



$$(- \partial_t D + \mathrm{rot}\, H = J, \quad \partial_t B + \mathrm{rot}\, E =$$

$\mathcal{N}$ □ $\overset{G}{\text{grad}}$ □ $\mathcal{E}$ □ $\overset{R}{\text{rot}}$ □ $\mathcal{F}$ □ $\overset{D}{\text{div}}$ □

Approximate representation of the field by degre
freedom assigned to both kinds of cells

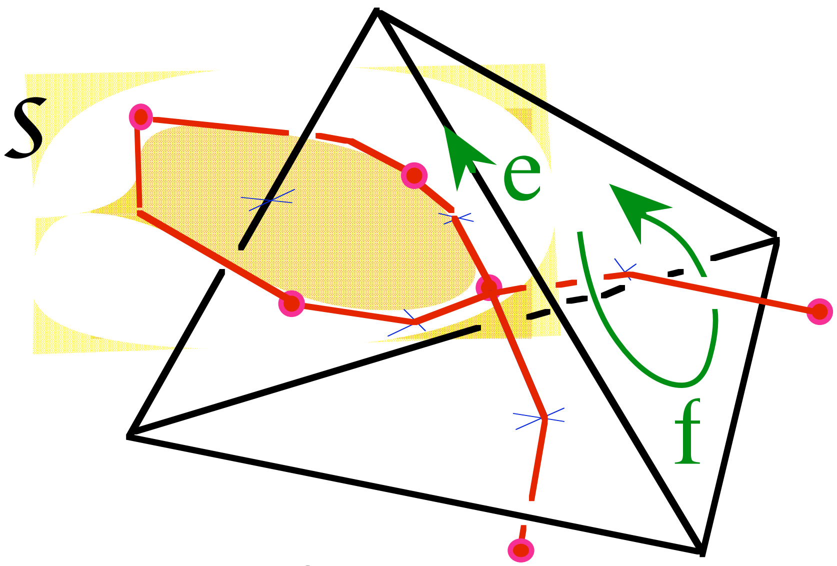
b *at faces*

h *at dual*

(i.e.

, **a**

at edges



here, $R_{fe} = -1$

d, j

at dual fa

xes

b = {b_f : f □ $\mathcal{F}$} ——□——▶ **h** = {h_f : f [

mf's

e = {e_e : e ∏ $\mathcal{E}$} ——⌈——▶ **d** = {d_e : e ⌈

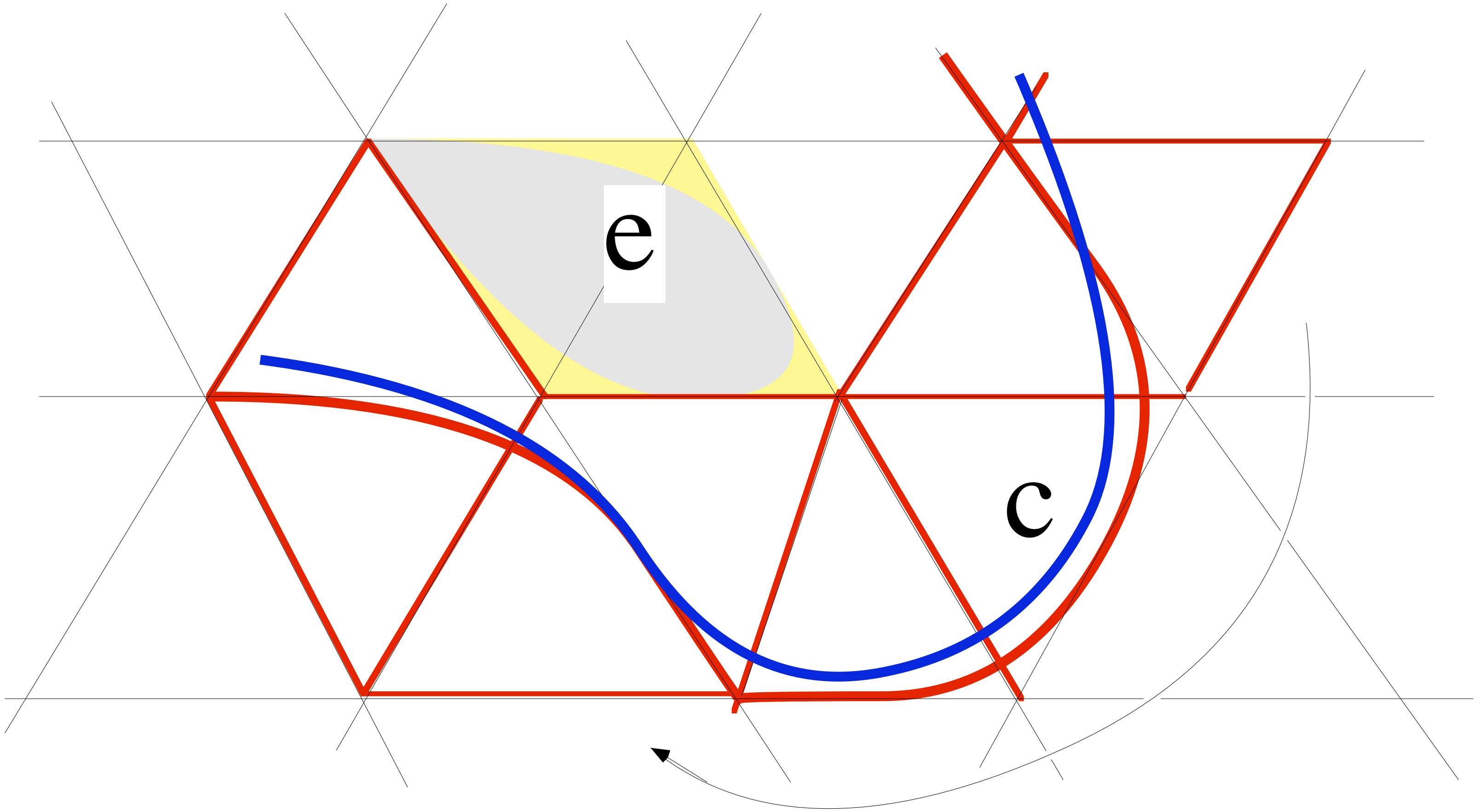
Whitney forms

Once obtained the cochains **b**, **e**, **h**, **d**,
what about the fields themselves?

or else:

Are there objects that would be to differential forms what finite elements are to fonctions, i.e., to 0-forms?

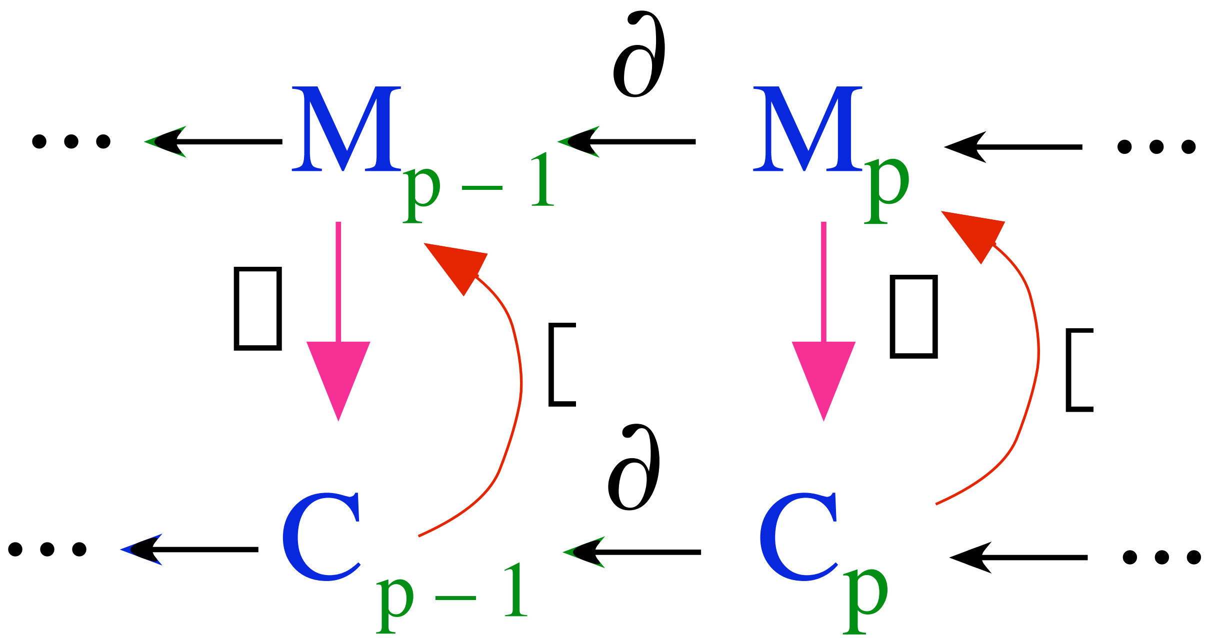
$\mathbf{e} = \sum_{a \in \mathcal{A}} \mathbf{e}_a \, w^a$ $\qquad$ $\mathbf{b} = \sum_{f \in \mathcal{F}} \mathbf{b}_f \, w^f$



If $c \sim \sum_{e \in \mathcal{E}} w^e(c)\, e$, then $\int_c a \sim \sum_e w^e(c)$

hence $\int_c a \sim \sum_e w^e(c)\, \mathbf{a}_e$, so $a \sim \sum_e \mathbf{a}_e\, w^e$

A chain map:

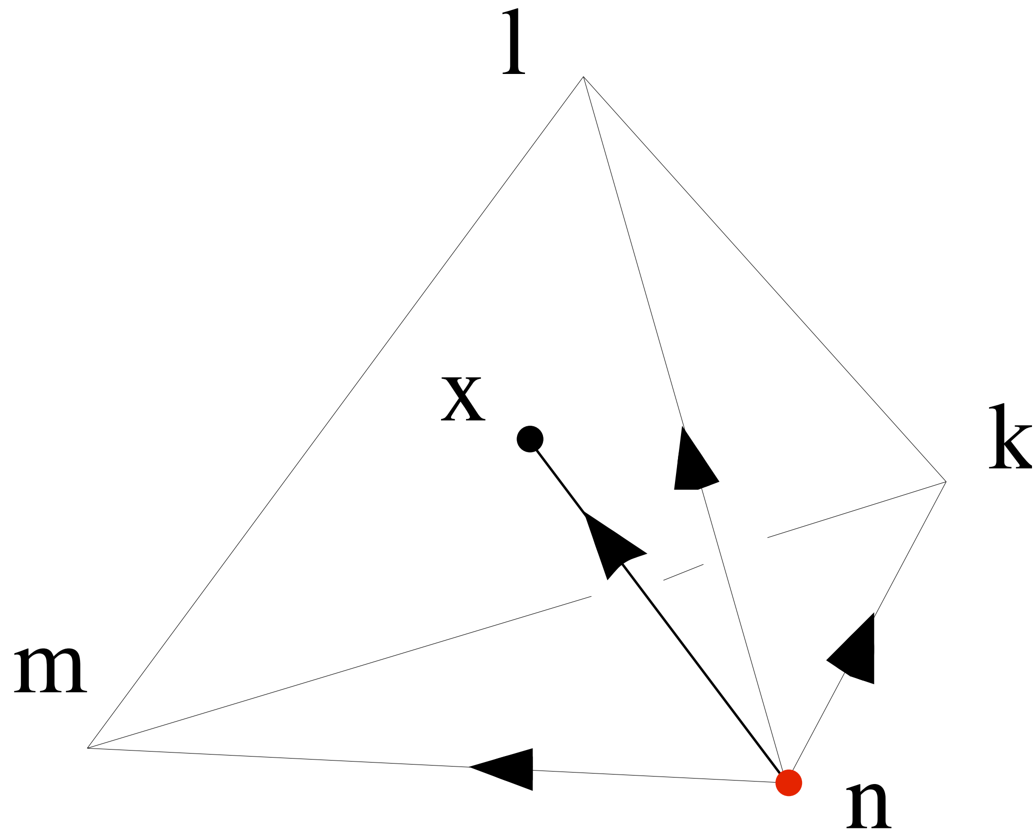


Thm.: □ *unique* □ *such that*

$\square\partial = \partial\square, \quad \square\square = 1, \quad \square(x) = \Sigma_{n\, \square\, \mathcal{N}}\, \square^{n}($

$\square(x) \approx \Sigma_{n}\, \square^{n}(x)\, \square_{n} \equiv \langle \Sigma_{n} \square^{n}(x)\, n\, ;\, \square \rangle = \langle \square(x)$

"Obviously", $\square(nx) = \sum_i \square^i(x)\, ni$, $i \square \{k, l, m, n$



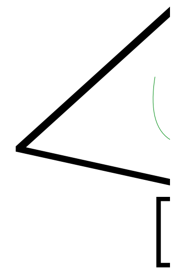
More formally:

$$\square\partial(nx) = \square(x) - \square(n) = \sum_i \square^i(x)(i - n) = \sum_i \square^i(x)$$

$$\square \quad \square(nx) = \sum_i \square^i(x)\, ni + \sum_{f \,\square\, \mathcal{F}} \square_f(x)\, \partial f \qquad i \,\square$$

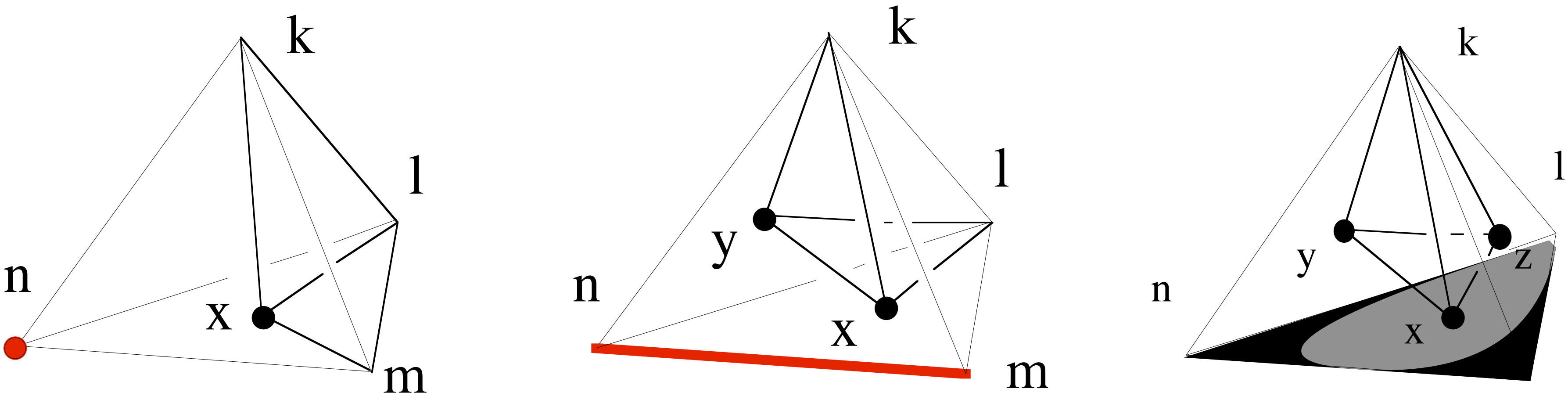
with $\square_f$ linear, but $[\square(nm) = nm] \;\square\; \square_f = 0.$

All gory details omitted,



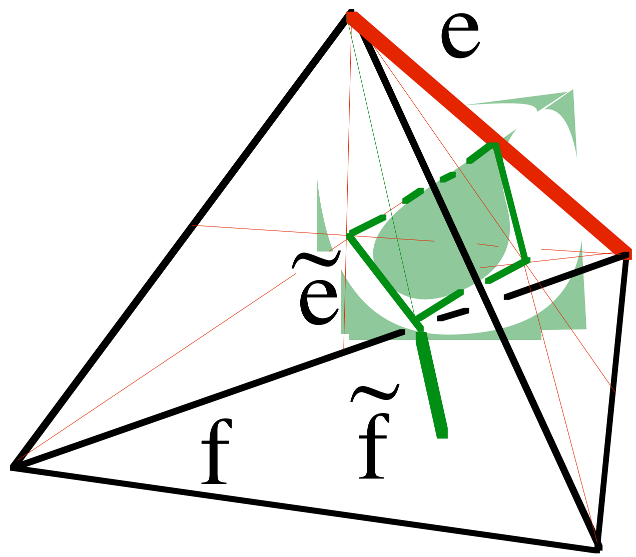
for a p-simplex s,

$$w^{s}(x) = \sum_{\square \square \{(p-1)\text{-simplices}\}} \partial^{s}_{\square} \, \square^{s-\square}(x)\, dw^{\square}$$



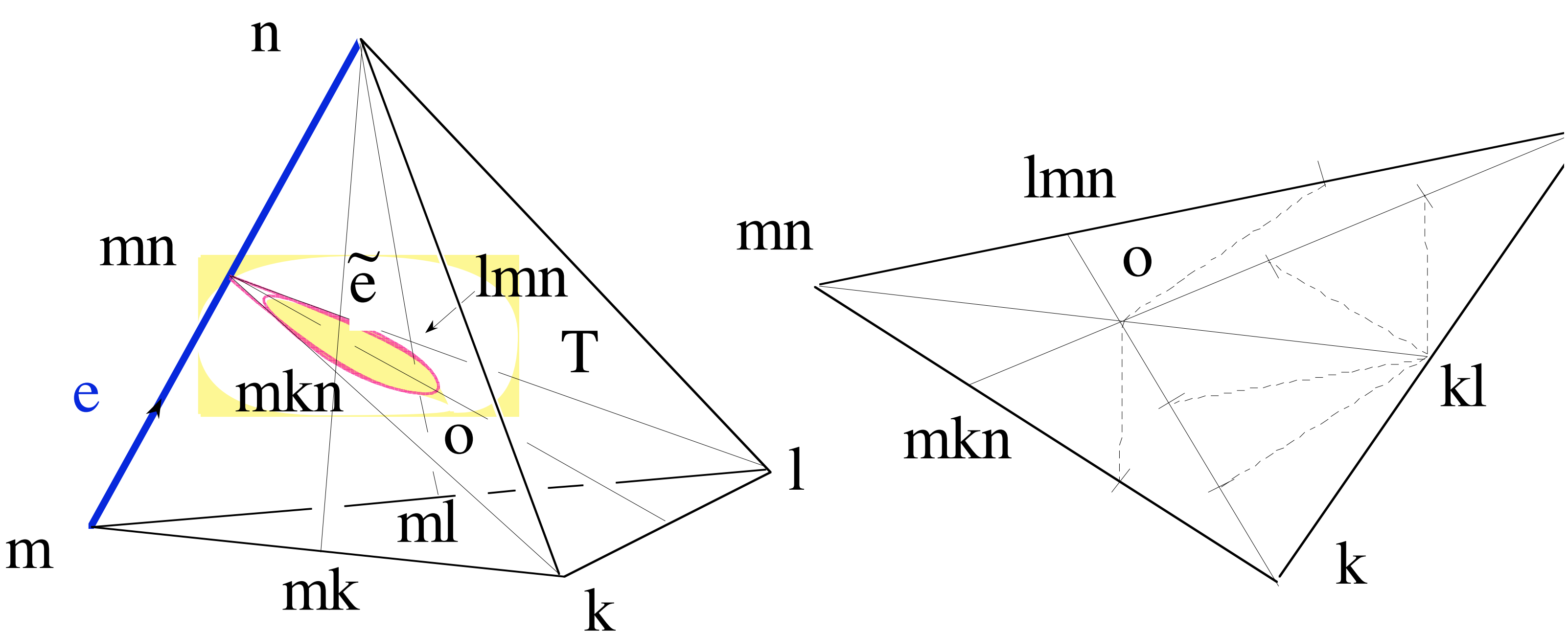
0 1 2

$$\Box_{\tilde{f}}\, w^e = \Box w^e \,\Box\, w^f = \Box_{\tilde{e}}\, w^f$$



So if b is constant, $\Box w^e \,\Box\, b = \Box_{\tilde{e}}\, b$, i.e., " $\Box w^e$

$$\square w^{e} = \tilde{e}, \quad \square w^{f} = \tilde{f}$$



$$\square_T \square w^{n} = \{k, l, m\}/3$$

$$\square_T w^{m} \square w^{n} - w^{m} \square w^{n} =$$

$$(\{k, l, m\}/3 + \{k, l, n\}/3)/4 = \tilde{e}$$

Whitney forms as a partition of un

- $\sum_{n} w^{n}(x) = 1 \quad \square\ x$

- $\sum_{e} w^{e}(x) \quad e = 1 \quad \square\ x$

  i.e., $\sum_{e} (v \cdot w^{e}(x))\, e = v \quad \square$

- $\sum_{f} w^{f}(x) \quad f = 1 \quad \square\ x$

etc.

Consequence: The mass matrix M

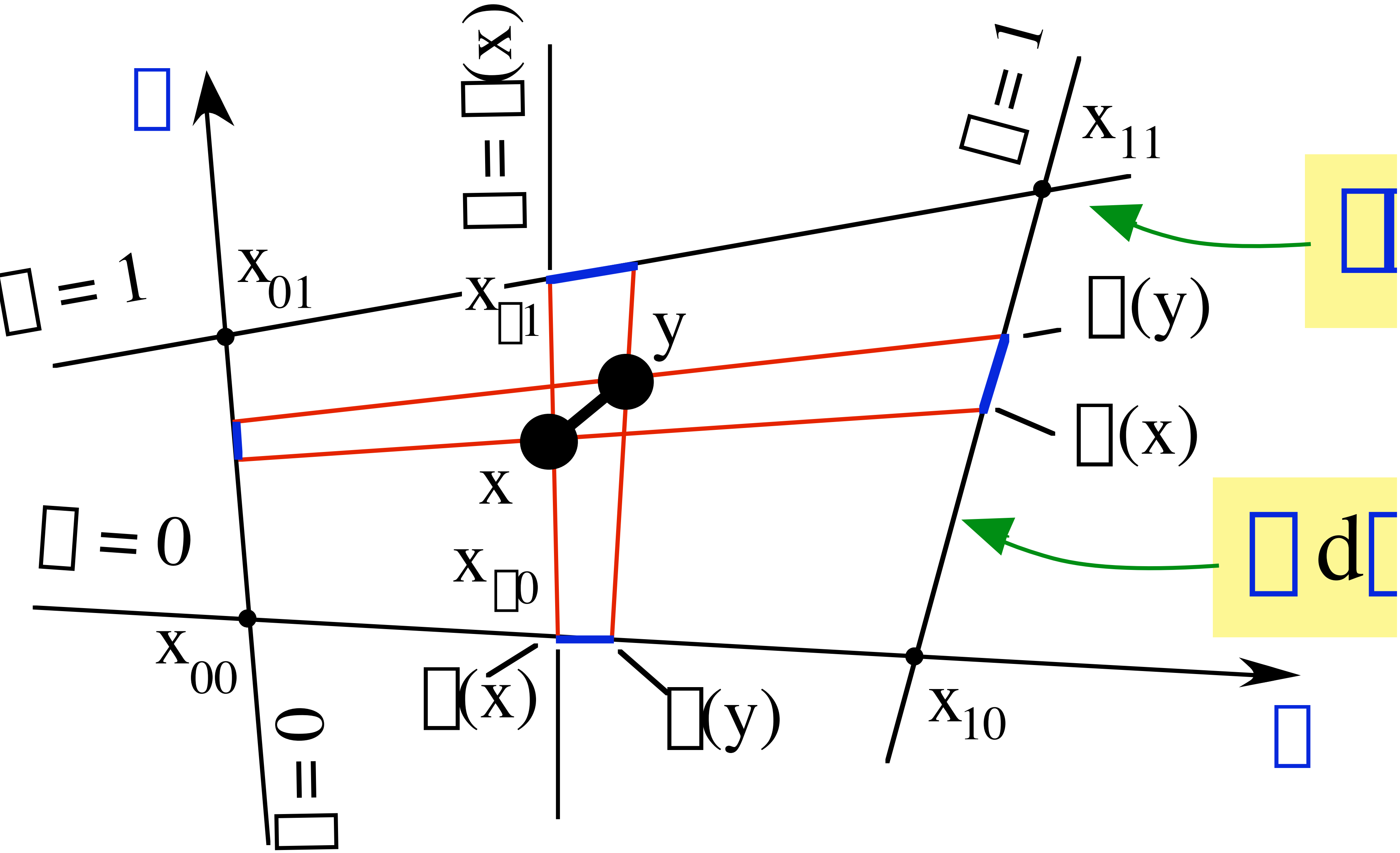
of edge elements ...

$$\Sigma_{e'}\ w^{e}(x) \cdot w^{e'}(x)\ e' = w^{e}(x)$$

$$\Sigma_{e'}\ [\int_D w^{e}(x) \cdot w^{e'}(x)\ dx]\ e' = \int_D\ w^{e}(x$$

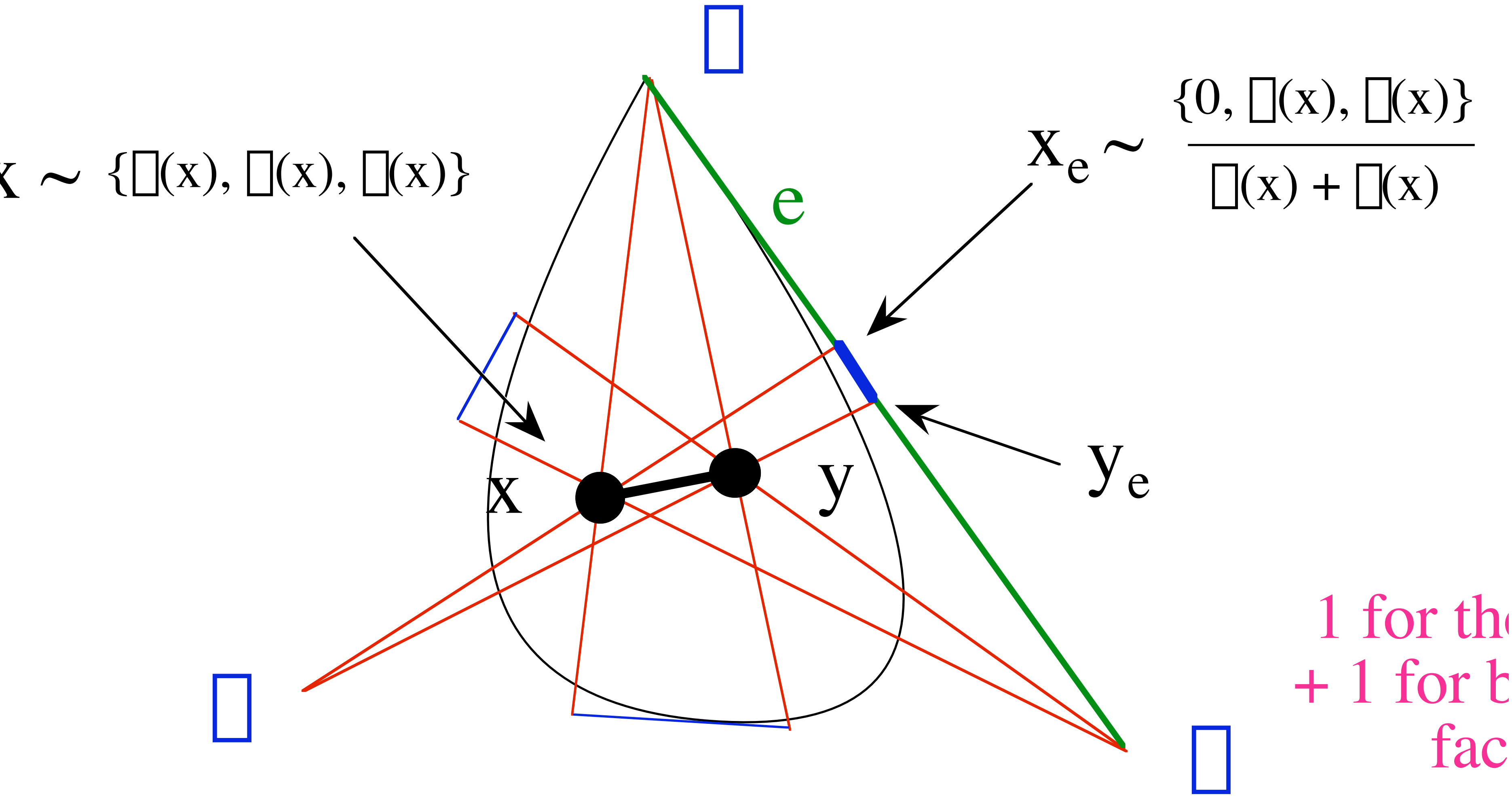
$$\Sigma_{e'}\ M^{ee'}\ e' = \int w^{e}(x) = \tilde{e}$$

... satisfies the consistency requireme



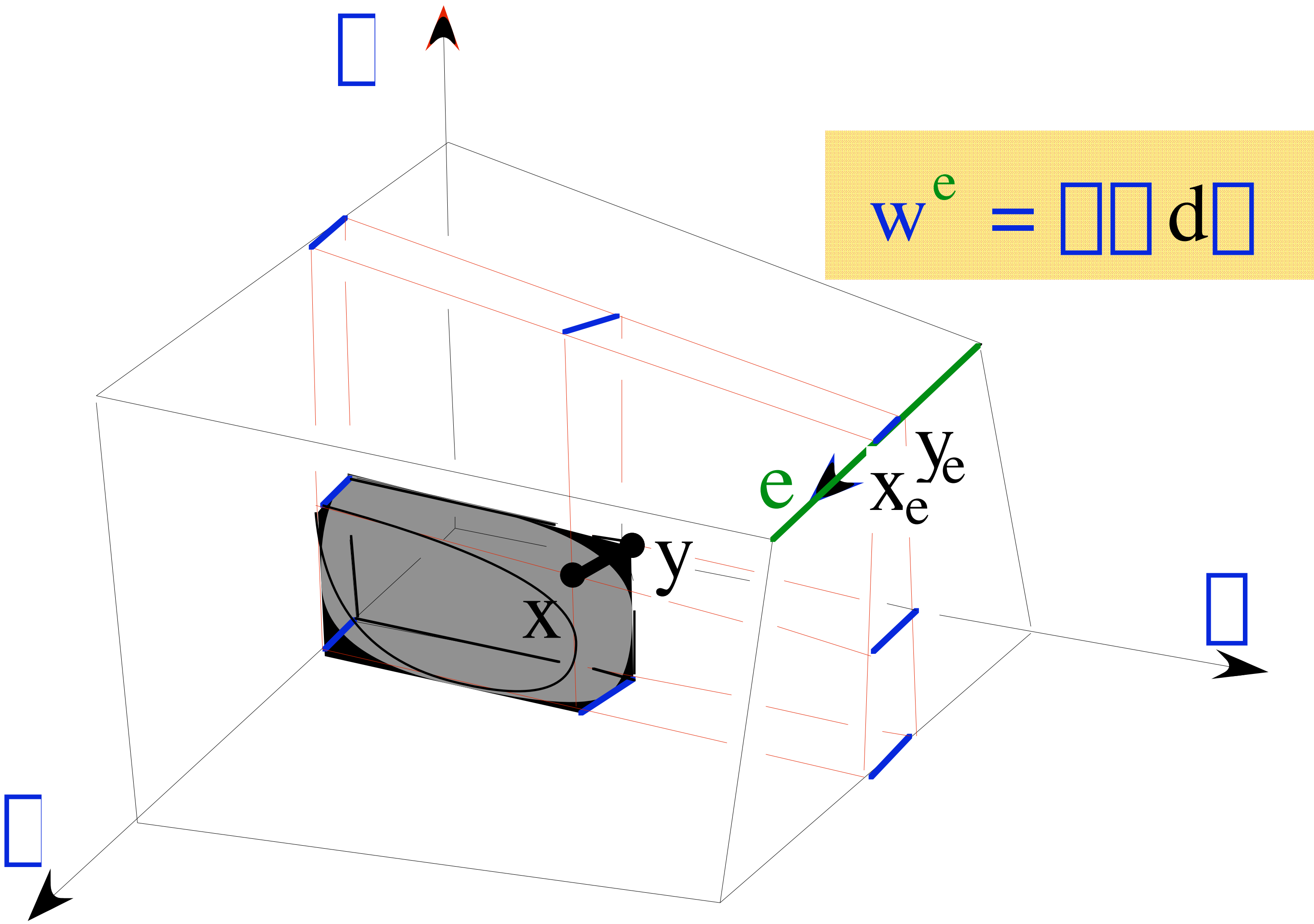
Weight of x w.r.t. x_{11}: □(x) □(x)

Weight of xy w.r.t. x_{10} x_{11}: $\frac{1}{2}$ [□(x) + □(y)](□(y)

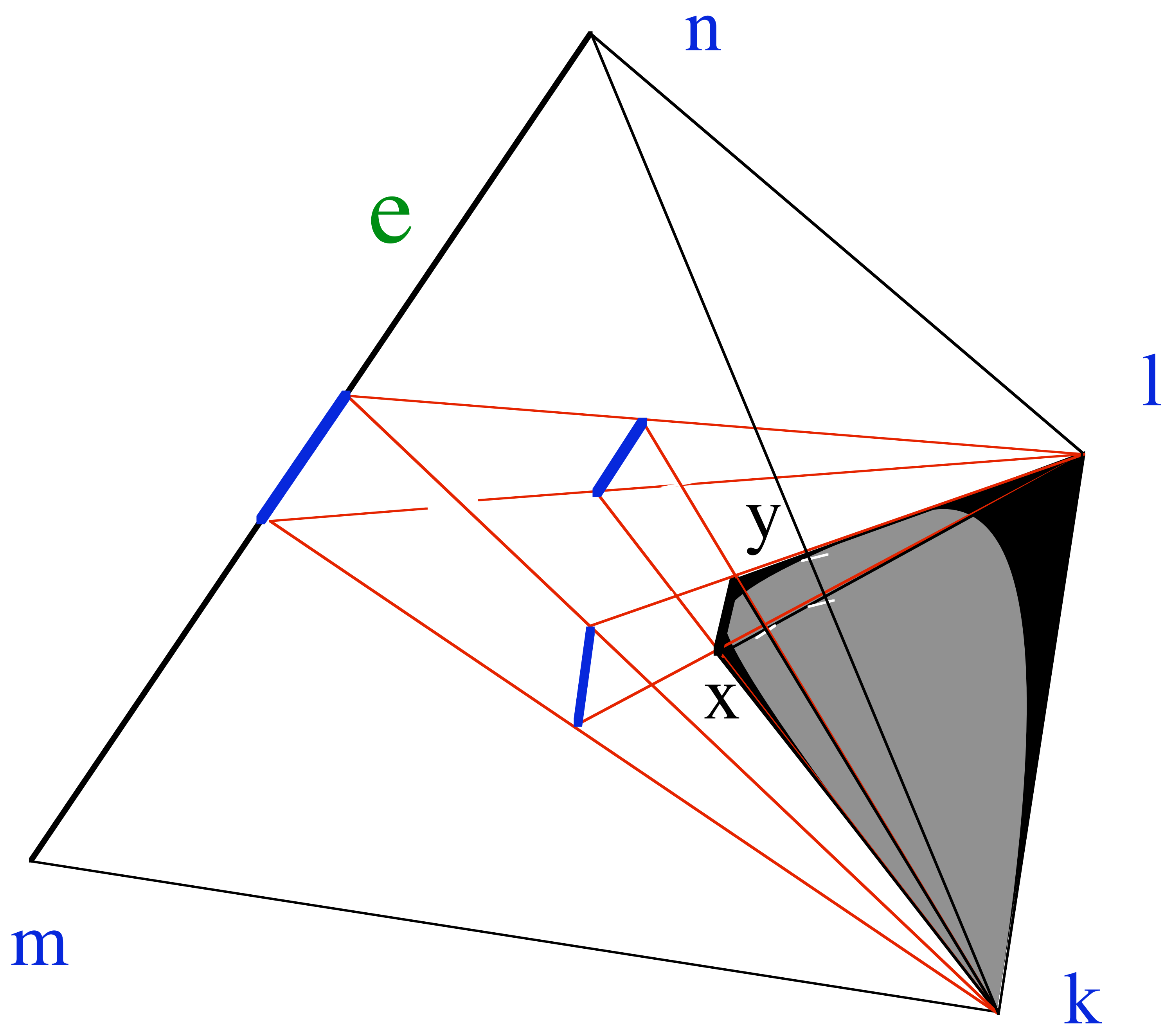


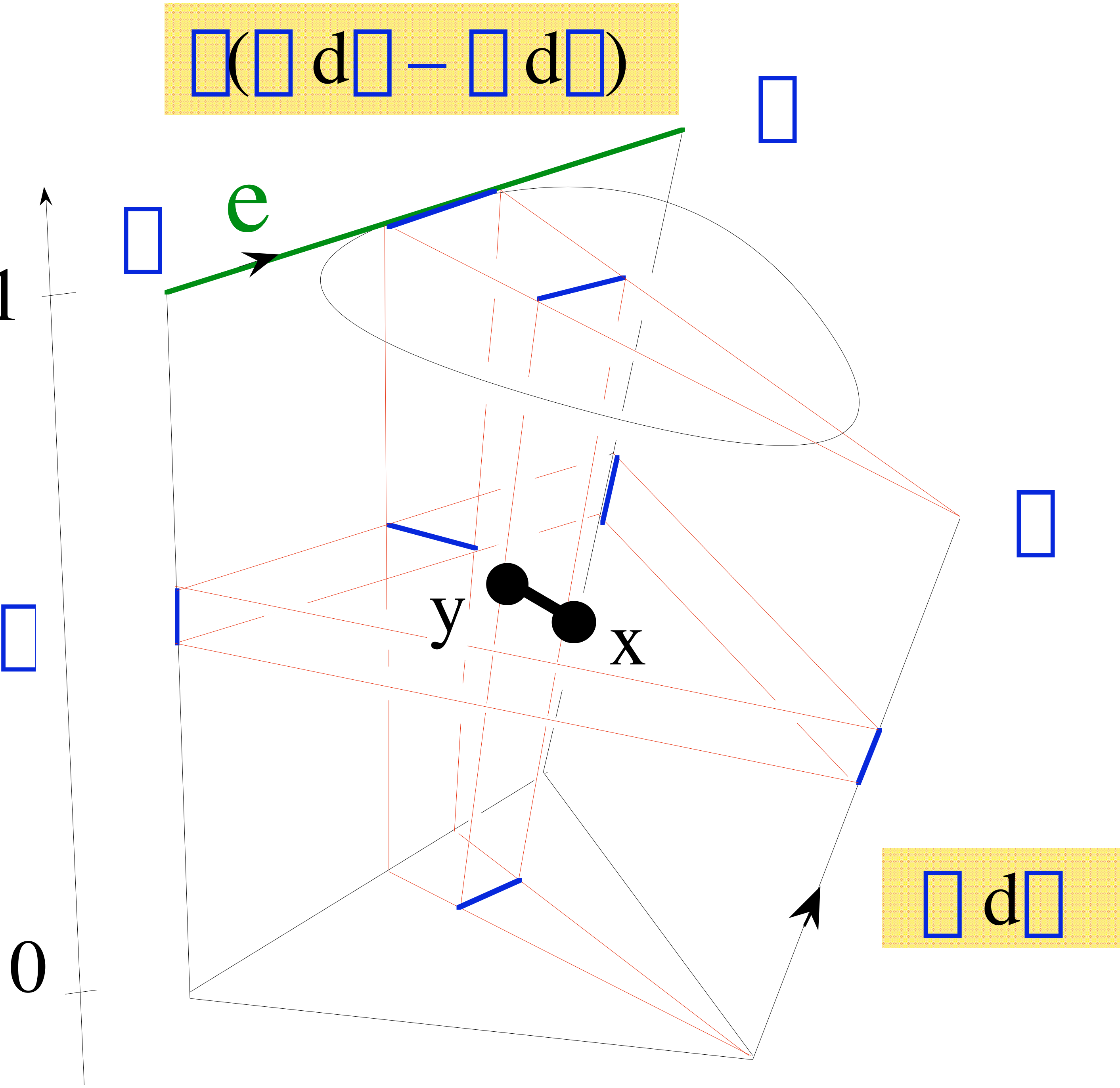
Weight of xy w.r.t. e is $\frac{x_e\, y_e}{x_\square\, x_\square}$ □ [(□ + □)($\frac{x + y}{2}$

which is (□d□ – □d□)(xy) indeed

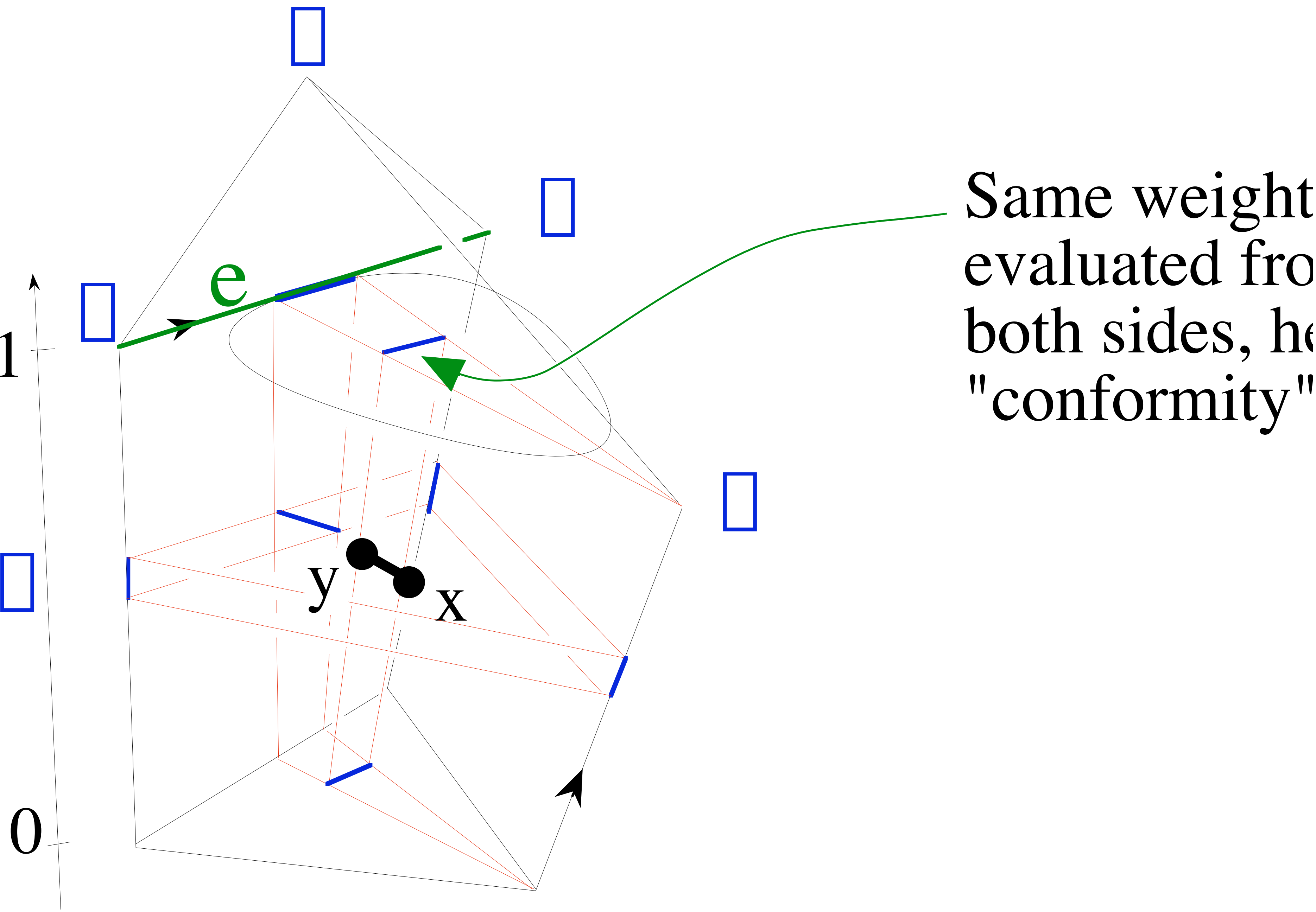


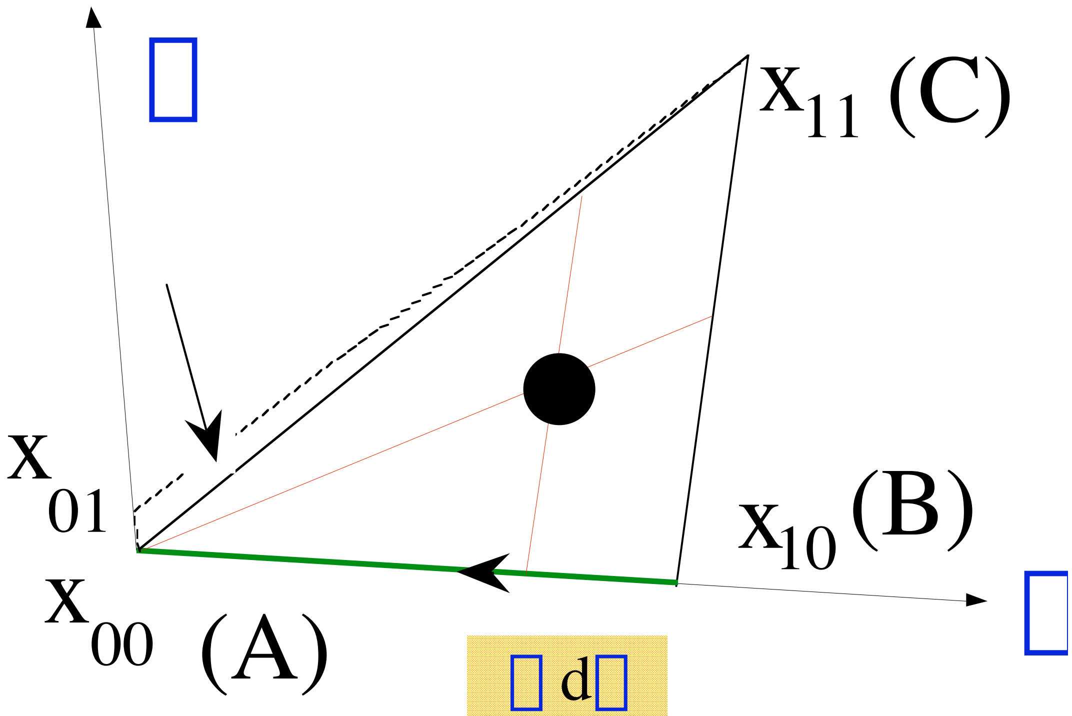
d□(xy) = $\frac{x_e y_e}{e}$. Weights □(x) and □(x)
gathered by projecting left then up


n
e
l
y
x
m
k



Simplicial
projection
in p dime
"isoparame
one in n –
other dime


Same weight
evaluated fro
both sides, he
"conformity"
e
y
x
1
0



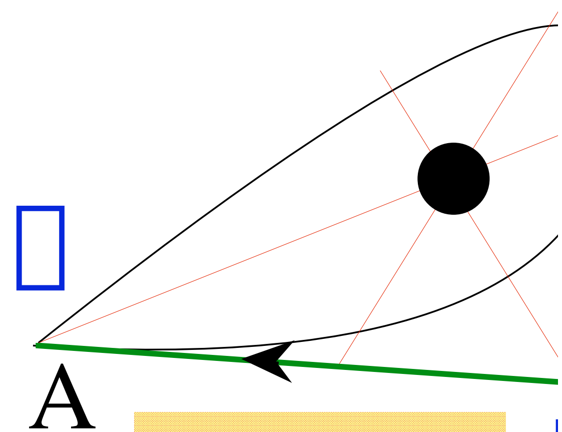
$$\frac{□\, d□}{1 - □}$$

Not the sam

$$□ = \frac{□}{□ + □}$$

$$□ = □ + □$$

$(□ - 1)d□$



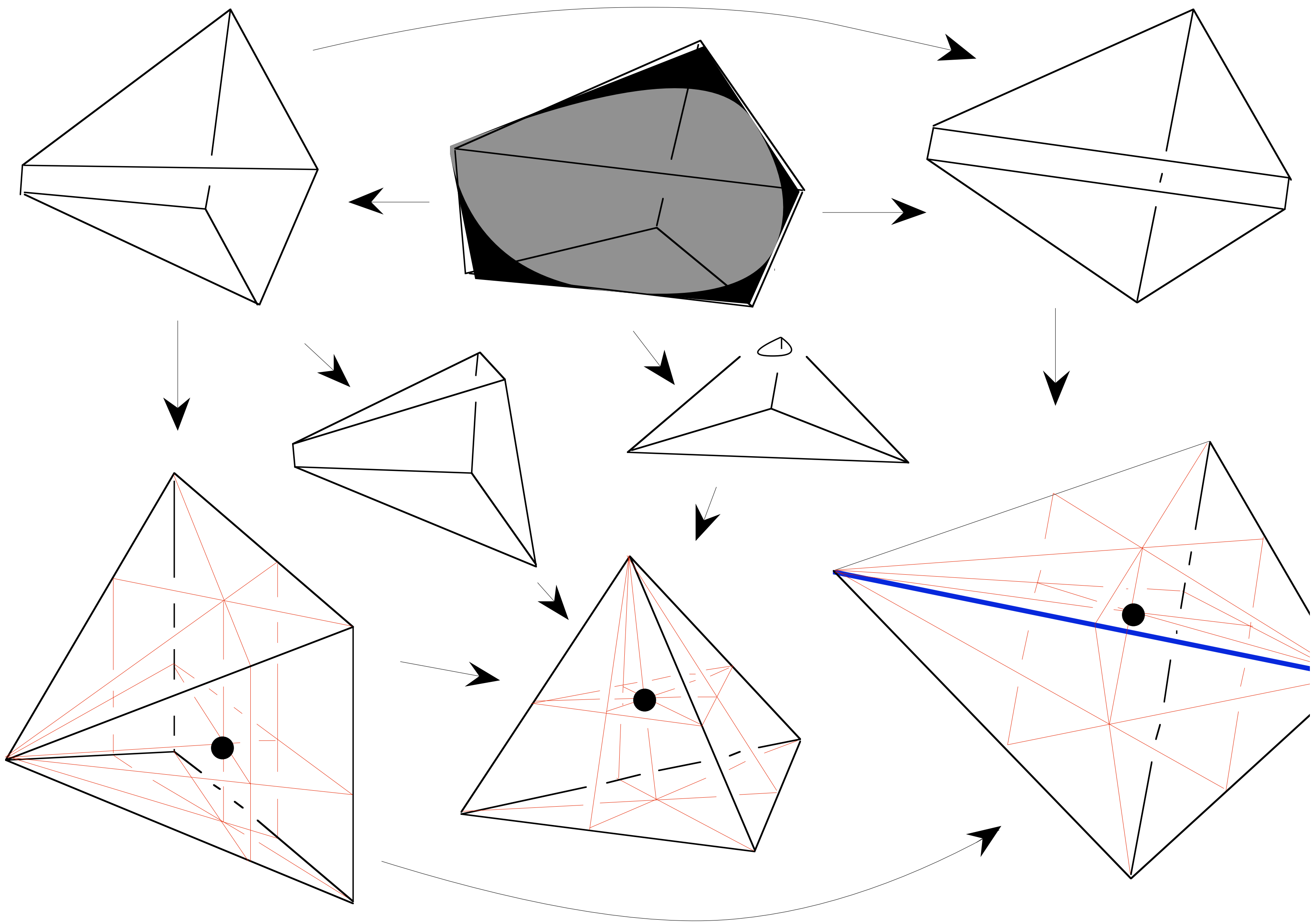
$□\, d□ - □\, d□$

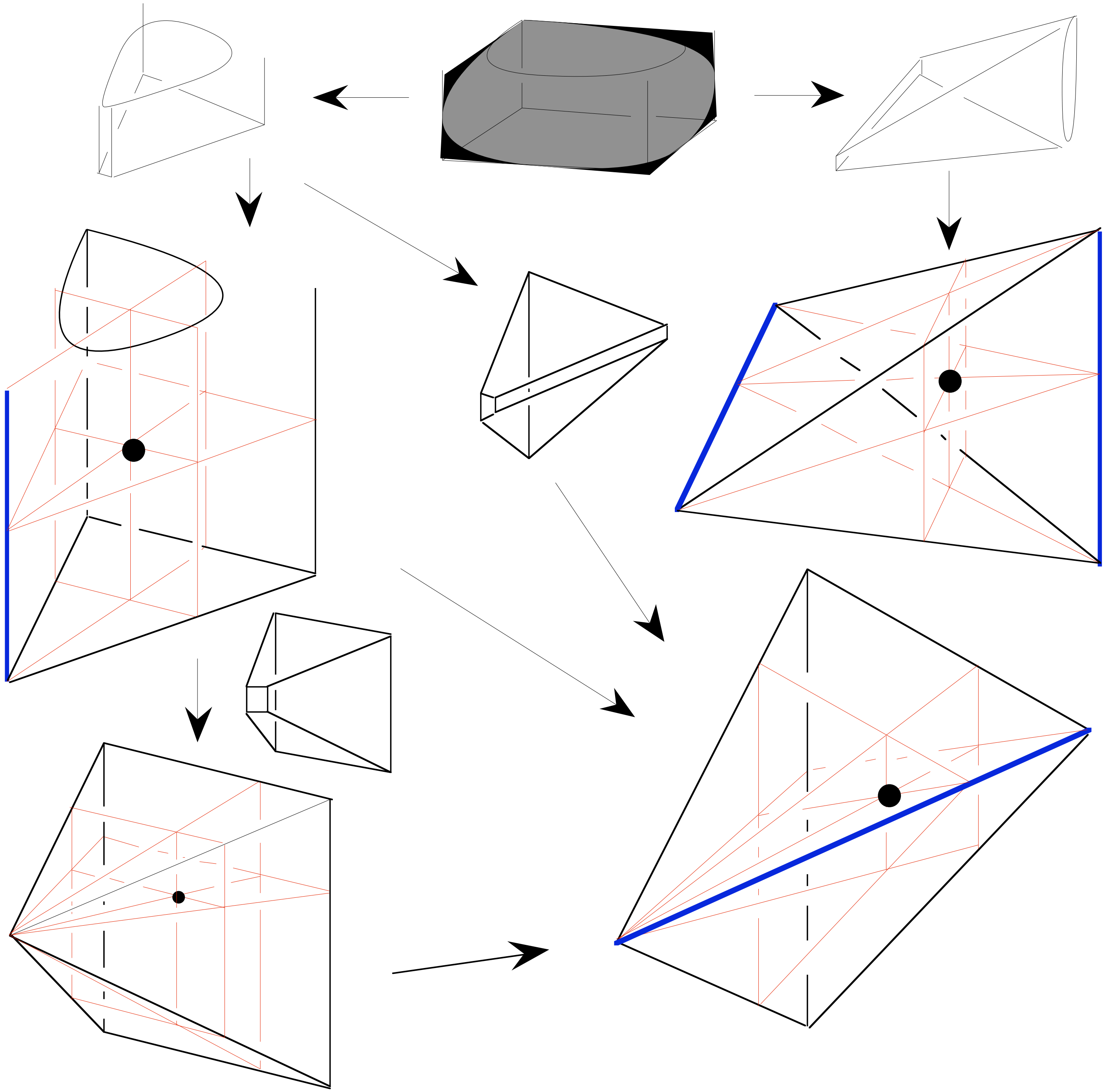
"Degeneracies": Theory

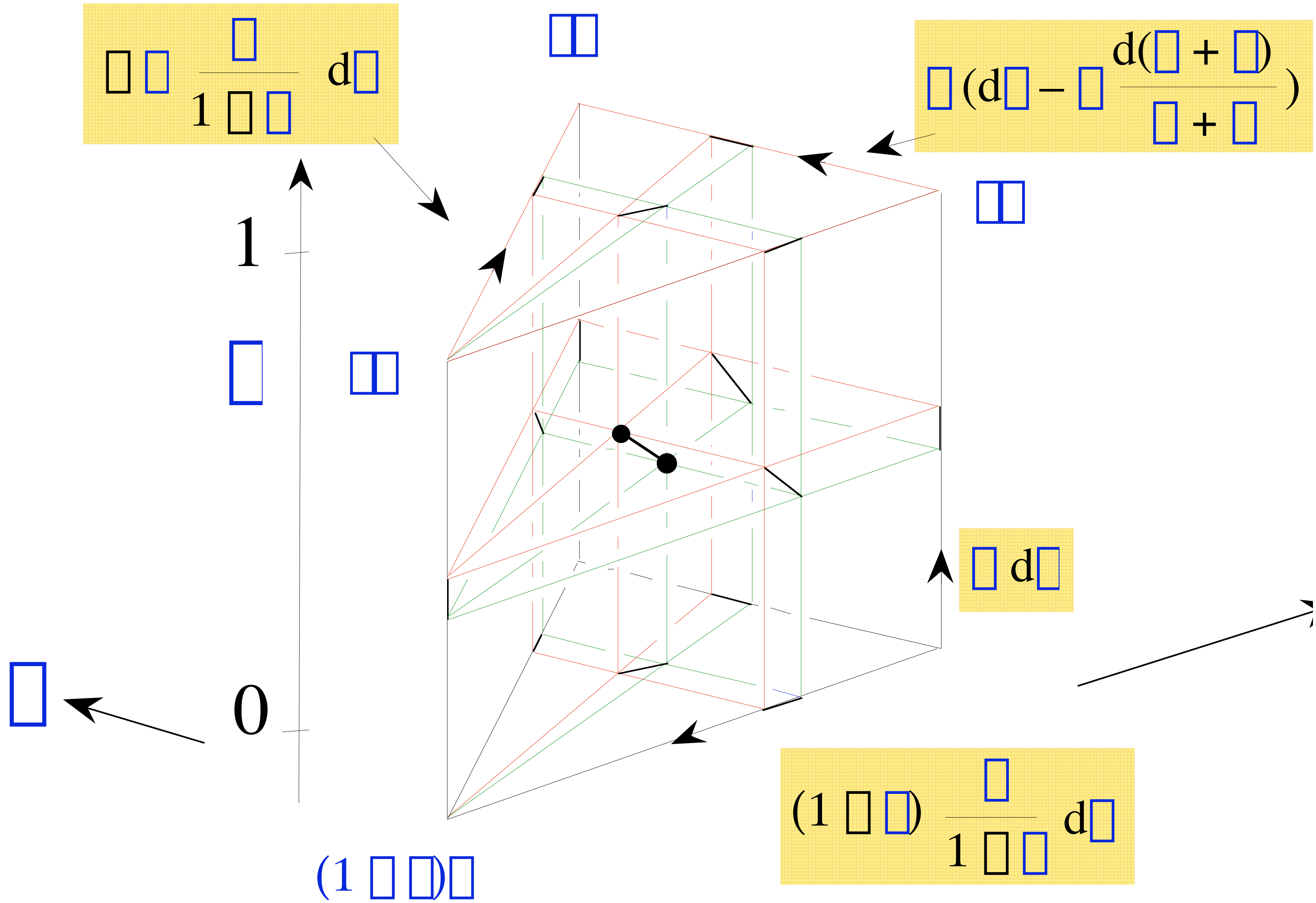
$$\begin{array}{ccccccc}
\cdots & \leftarrow & M_{p-1} & \xleftarrow{\partial} & M_p & \leftarrow & \cdots \\
 & & \square \downarrow & & \downarrow \square & & \\
\cdots & \leftarrow & C_{p-1} & \xleftarrow{\partial} & C_p & \leftarrow & \cdots \\
 & & \square \downarrow & & \downarrow \square & & \\
\cdots & \leftarrow & C^{\wedge}_{p-1} & \xleftarrow{\partial} & C^{\wedge}_p & \leftarrow & \cdots
\end{array}$$

(chain m

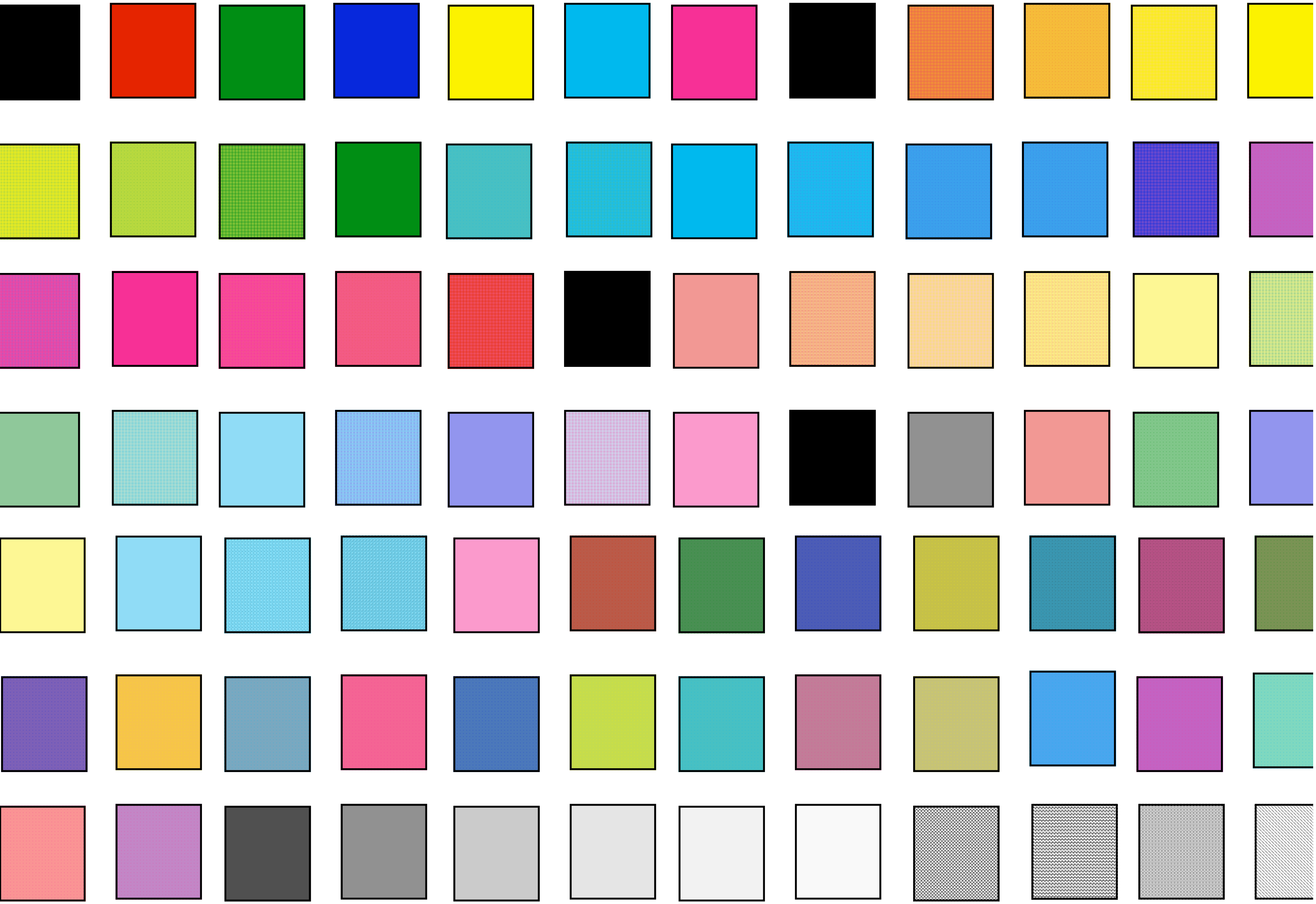
$$\text{Whitney}(\square(s)) = \sum \text{Whitney}(\square^{-1}(s))$$

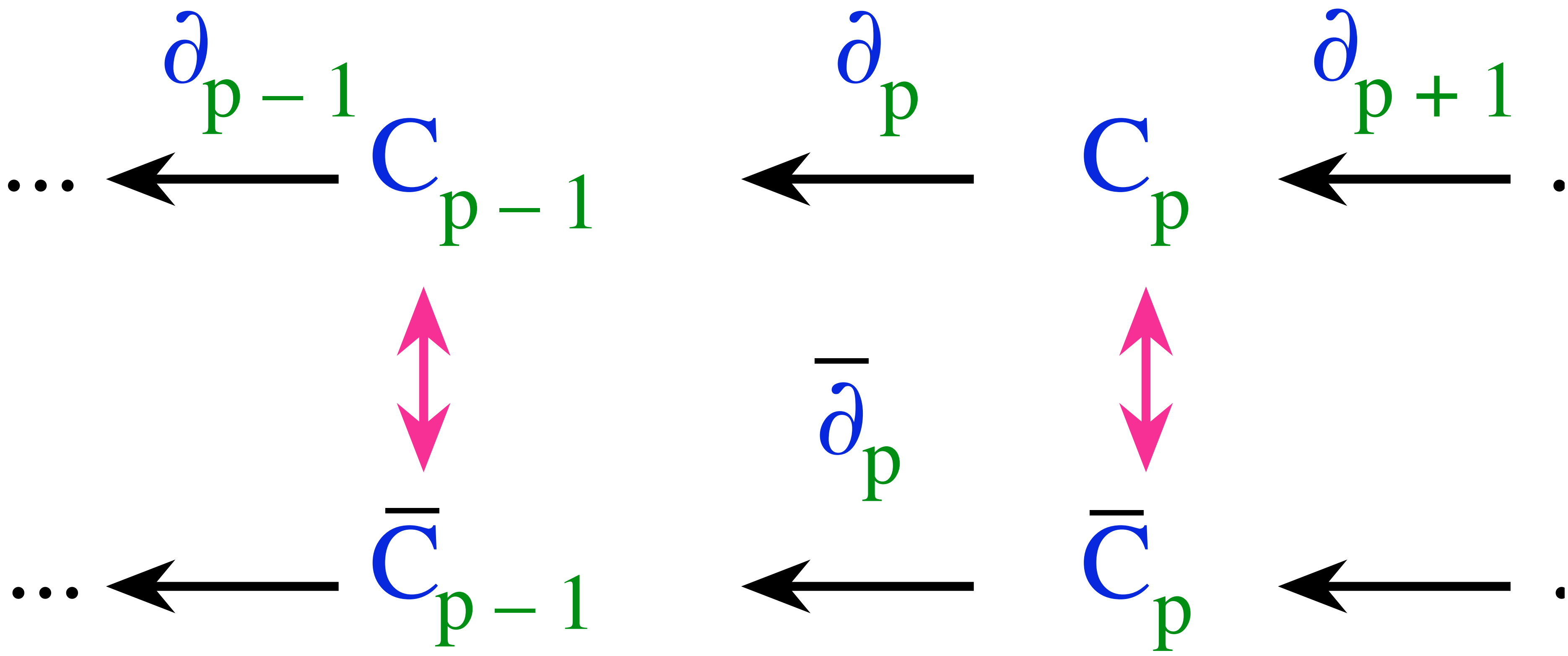




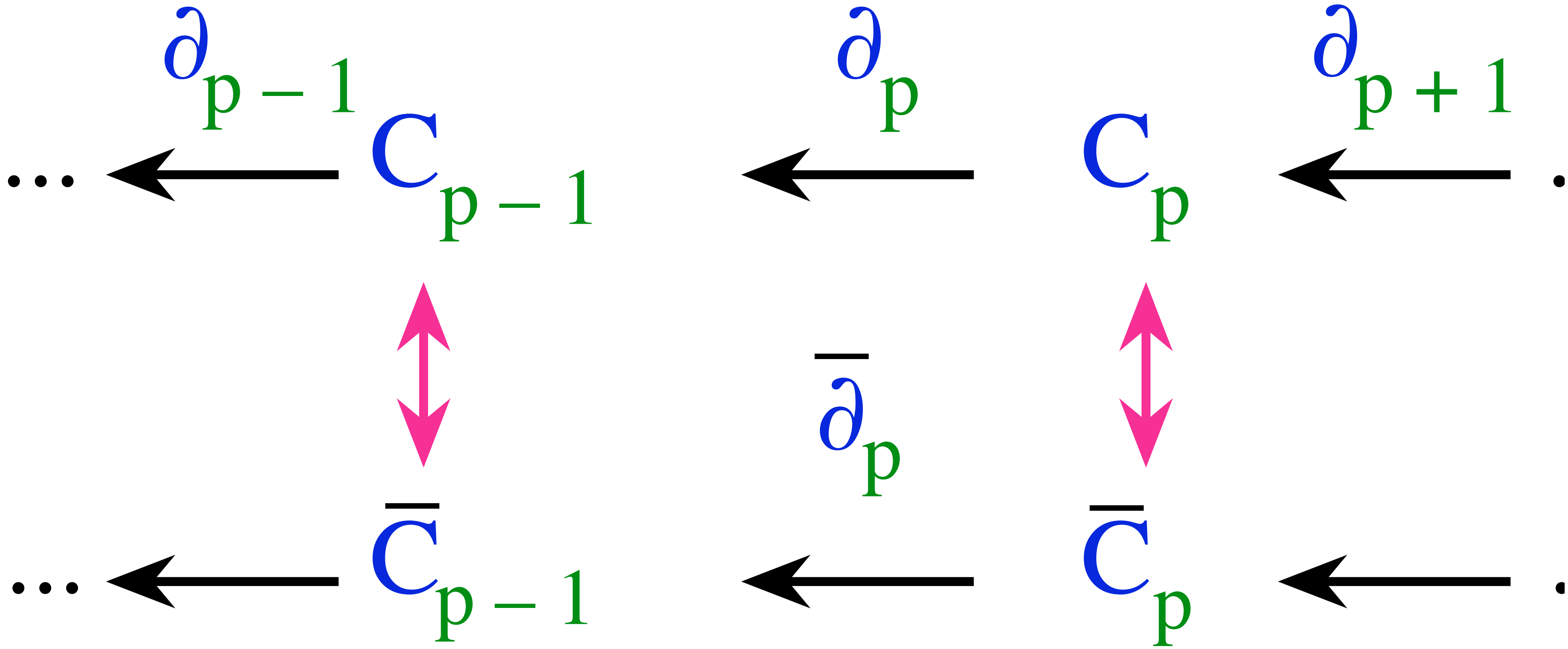


Old coordinates □, □, □. New ones, □, □, □,





∂p − 1
∂p
∂p + 1
Cp − 1
Cp
∂p
Cp − 1
Cp



∂p − 1
∂p
∂p + 1
Cp − 1
Cp
∂p
Cp − 1
Cp