Discrete differential geometry: Surfaces made from Circles

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with help of
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Papers

- A.I. Bobenko, T. Hoffmann, B. Springborn. Minimal surfaces from circle patterns: Geometry from combinatorics. `arXiv:math.DG/0305184`
- A.I Bobenko, B. Springborn. Variational principles for circle patterns and Koebe's theorem, Trans. AMS **356:2** (2003) 659-689
- A.I. Bobenko, D. Matthes, Yu.B. Suris, Discrete and smooth orthogonal systems: C^∞-approximation, Internat. Math. Research Notices **2003:45**, 2415-2459
- A.I. Bobenko, U. Pinkall, Discretization of Surfaces and Integrable Systems, In: A.I. Bobenko, R. Seiler (eds.) "Discrete Integrable Geometry and Physics", Oxford University Press (1999), 3-58
- A.I. Bobenko, U. Pinkall, Discrete Isothermic Surfaces, J. reine angew. Math. **475** (1996) 187-208
- S.I. Agafonov, A.I. Bobenko, Discrete Z^a and Painleve Equations, International Math. Research Notices **2000:4**, 165-193

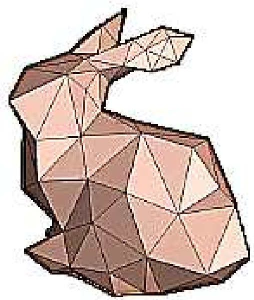
Discrete differential geometry

Aim: Development of discrete equivalents of the geometric notions and methods of differential geometry. The latter appears then as a limit of refinements of the discretization.

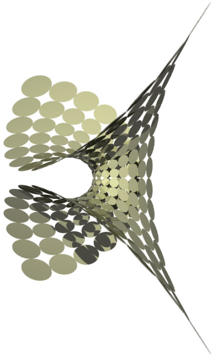
- new geometric objects in Discrete Geometry
- new methods (difference equations, geometric understanding of integrability)
- deep understanding of smooth theory (unification of surfaces and their transformations)
- solution of problems in Differential Geometry

Discrete surfaces

Triangulated (Euclidean)



Circular (Möbius)

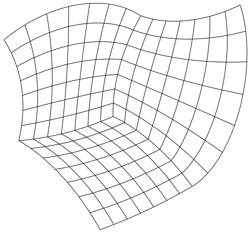


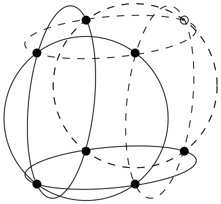
Natural in Möbius geometry:

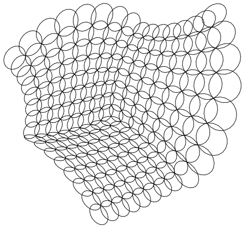
- curvature line parametrized surfaces
- conformally parametrized surfaces
- (triply)-orthogonal coordinate systems

Discrete orthogonal coordinate systems

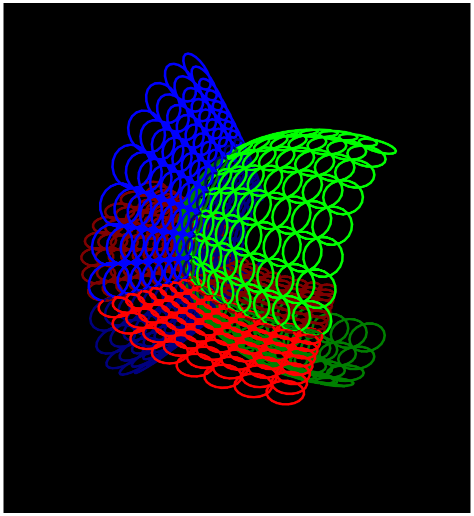
Dupin Theorem. Coordinate surfaces of a triply orthogonal coordinate system intersect along their common curvature lines







- Circular lattices as discrete curvature line parametrization [Martin et al., Nutbourne '86]
- Discrete orthogonal coordinate systems [B.'96]
- Cauchy problem based on Miguel's theorem [Cieśliński, Doliwa, Santini '97]
- Convergence with all derivatives (C^∞-convergence) [B./Matthes/Suris '03] movie.gif



Discrete conformal energy

$$W = \sum_k \beta_k - 2\pi$$

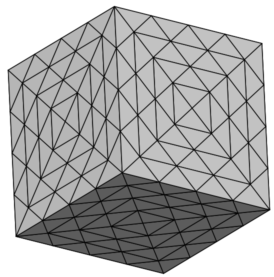
β_i are external angles between the circumscribed circles.

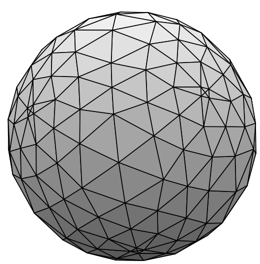
$W \geq 0$ and $W = 0$ iff the vertex and all its neighbors lie on a sphere. Minimizing of W makes the surface as round as possible. Analogue of the Willmore energy

$$W = \int (k_1 - k_2)^2$$

- conformal
- $W \geq 0$ and $W = 0$ iff the round sphere
- $W = \int H^2 - \int K$, on compact surfaces $\int K$ topological invariant
- related to elastic energy

Minimizing discrete conformal energy

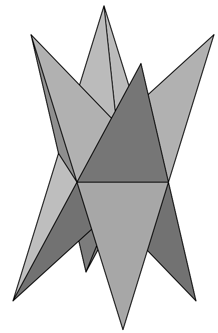


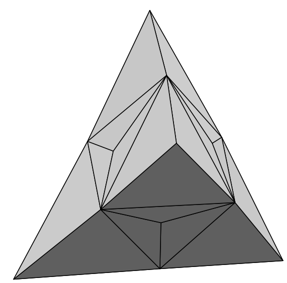


Triangulation:

"spherical", $W = 0$

sphere.gif

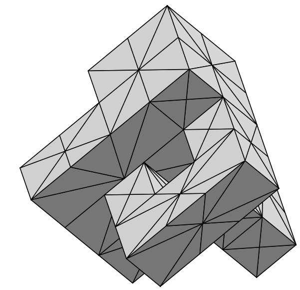


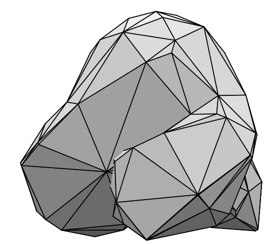


Triangulation:

"non-spherical", $W > 0$

nonsphere.gif



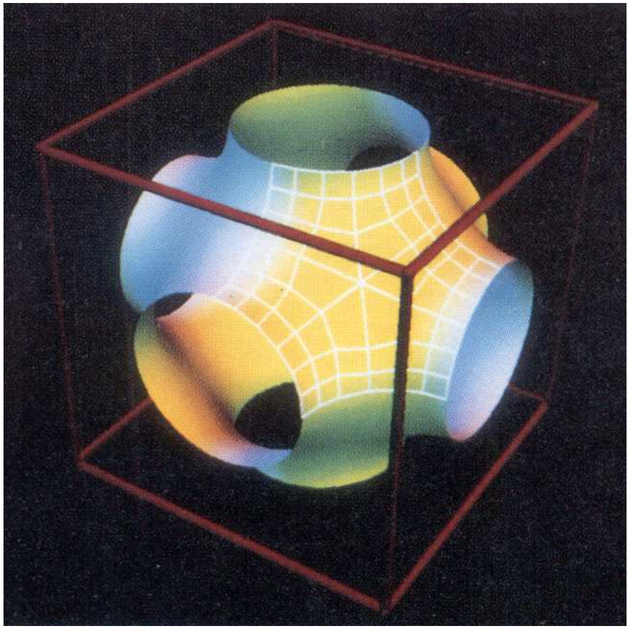


discrete Boy surface

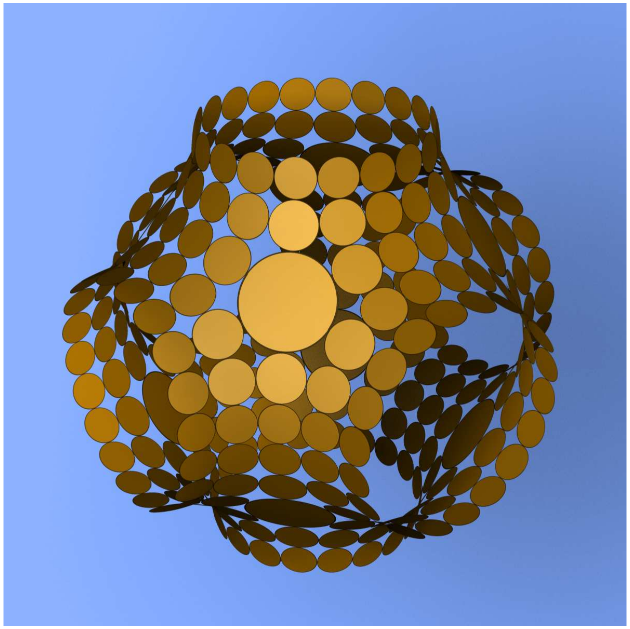
(projective plane)

boy.gif

Minimal surface: Schwarz' P-surface



continuous*

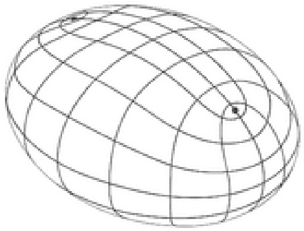


discrete

* from: Dierkes et al. *Minimal Surfaces I*. Springer 1992.

Isothermic surfaces

continuous



[Hilbert/Cohn-Vossen]

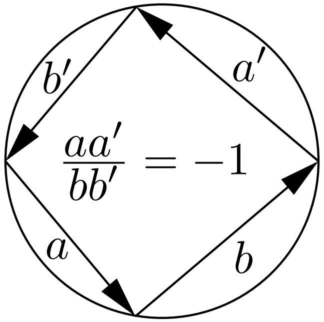
Definition. A surface in 3-space is called *isothermic* if it admits conformal curvature line coordinates.

- Definition is Moebius invariant.
- Curvature lines divide the surface into infinitesimal squares.

Examples: surfaces of revolution, quadrics, constant mean curvature surfaces, minimal surfaces.

Isothermic surfaces

discrete



Definition. A polyhedral surface in 3-space is called *discrete isothermic* if all faces are conformal squares, i. e. planar with cross ratio -1. [B./Pinkall '96]

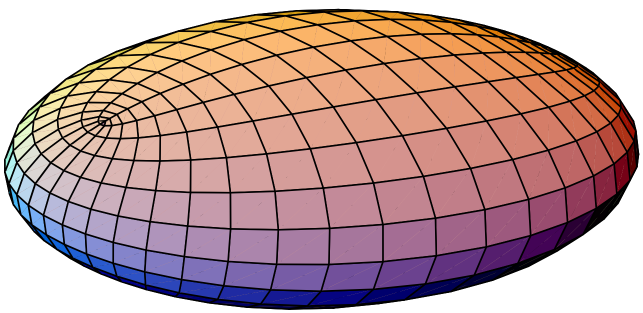
- Definition is Moebius invariant.
- 'Curvature lines' divide the surface into conformal squares.

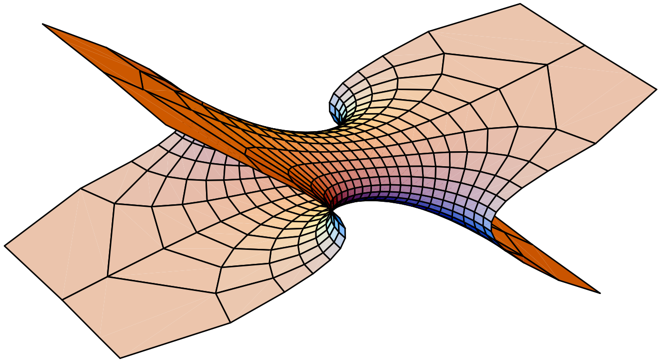
Duality for isothermic surfaces

continuous

Definition/Theorem. If $f : \mathbb{R}^2 \supset D \to \mathbb{R}^3$ is an isothermic immersion, then the *dual isothermic* immersion is defined by

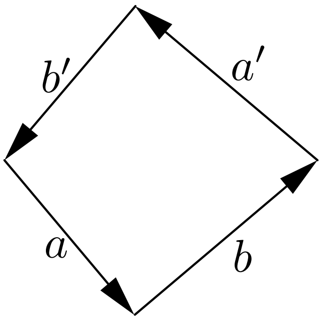
$$df^* = \frac{f_x}{\|f_x\|^2}\, dx - \frac{f_y}{\|f_y\|^2}\, dy.$$





Duality for isothermic surfaces

discrete



Proposition. *Suppose* $a, b, a', b' \in \mathbb{C}$ *with*

$$a + b + a' + b' = 0 \quad \textit{and} \quad \frac{aa'}{bb'} = -1$$

and let

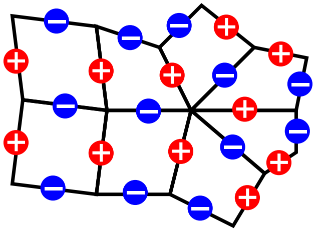
$$a^* = \frac{1}{\overline{a}}, \quad a'^* = \frac{1}{\overline{a}'}, \quad b^* = -\frac{1}{\overline{b}}, \quad b'^* = -\frac{1}{\overline{b'}}.$$

Then

$$a^* + b^* + a'^* + b'^* = 0 \quad \textit{and} \quad \frac{a^*a'^*}{b^*b'^*} = -1.$$

$$\Downarrow$$

Can define *duality for discrete isothermic surfaces* if edges may be labeled '+' and '−' appropriately.



Minimal surfaces

- Minimal surfaces are isothermic.
- Isothermic F is minimal. $\iff$ F^* contained in a sphere. (It's the Gauss map.)

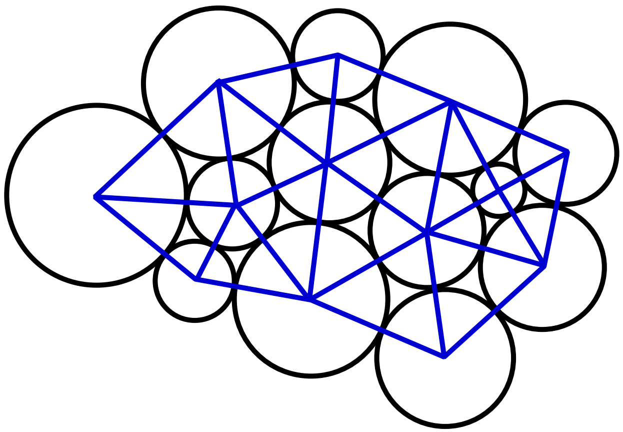
A way to construct minimal surfaces:

conformally parametrized sphere $\xrightarrow{\text{dualize}}$ minimal surface

Idea:

conformally parametrized discrete sphere $\xrightarrow{\text{dualize}}$ discrete minimal surface

Circle packings

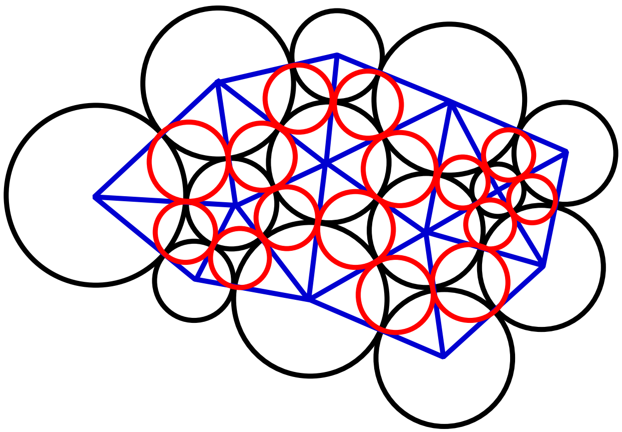


circle packing	$\longleftrightarrow$	triangulation
circles	$\longleftrightarrow$	vertices
touching circles	$\longleftrightarrow$	edges

Koebe's Theorem (1936). *To every triangulation of the sphere there corresponds a circle packing. It is unique up to Moebius transformations.*

> "Auf diesen Schließungssatz bzw. einen damit zusammenhängenden merkwürdigen Polyedersatz beabsichtige ich in einer besonderen Note zurückzukommen, die ich der Preuß. Akademie der Wissenschaften überreichen will."

Circle packings



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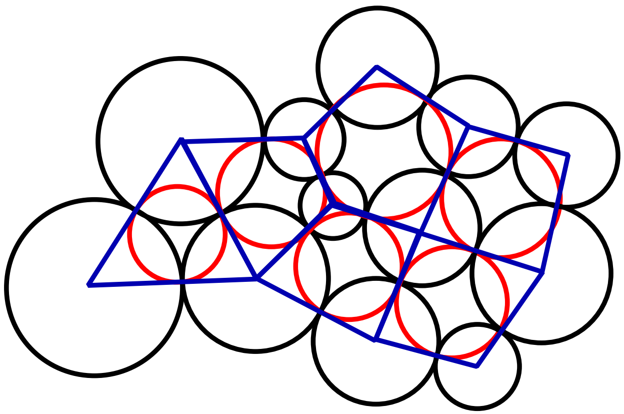
Orthogonal circle patterns



orthogonal circle pattern	$\longleftrightarrow$	polytopal cell decomposition
red circles	$\longleftrightarrow$	faces
black circles	$\longleftrightarrow$	vertices

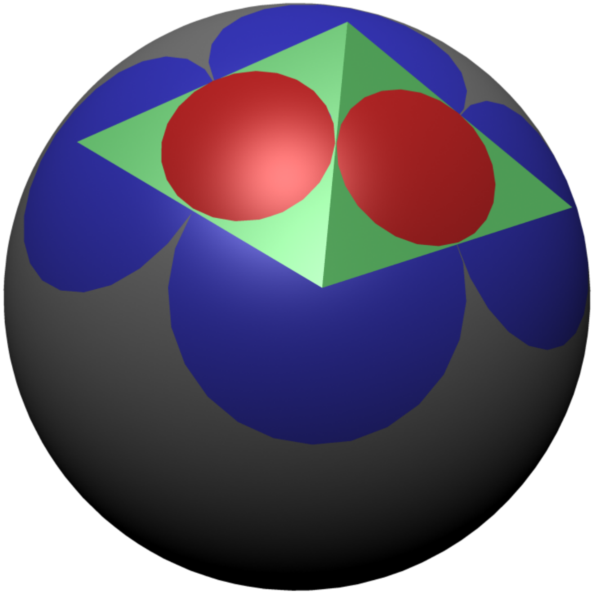
Red circles intersect black circles orthogonally.

Theorem. *To every polytopal cell decomposition of the sphere there corresponds an orthogonal circle pattern. It is unique up to Moebius transformations.*

Schramm '92 (more general result).

Brightwell/Scheinerman '93 (proof à la Thurston).

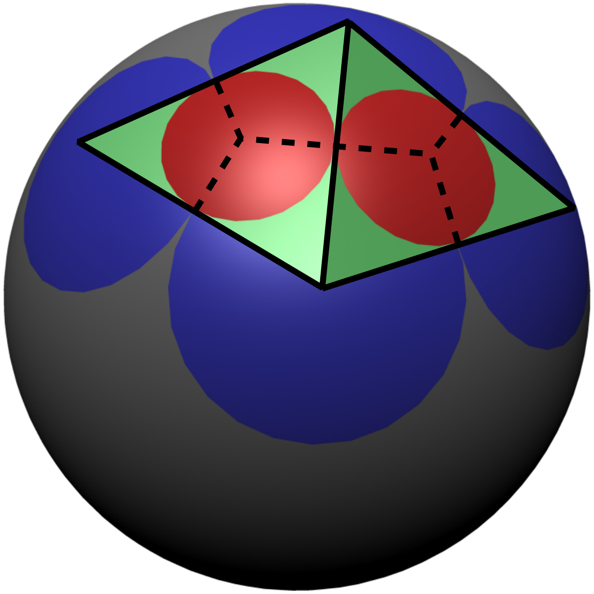
Koebe polyhedra

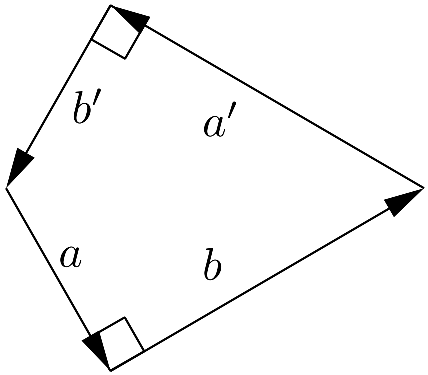


Theorem. *Every polytopal cell decomposition of the sphere can be realized by a polyhedron with edges tangent to the sphere. This realization is unique up to projective transformations which fix the sphere.*

There is a simultaneous representation of the dual cell decomposition with orthogonanally intersecting edges.

Koebe polyhedra



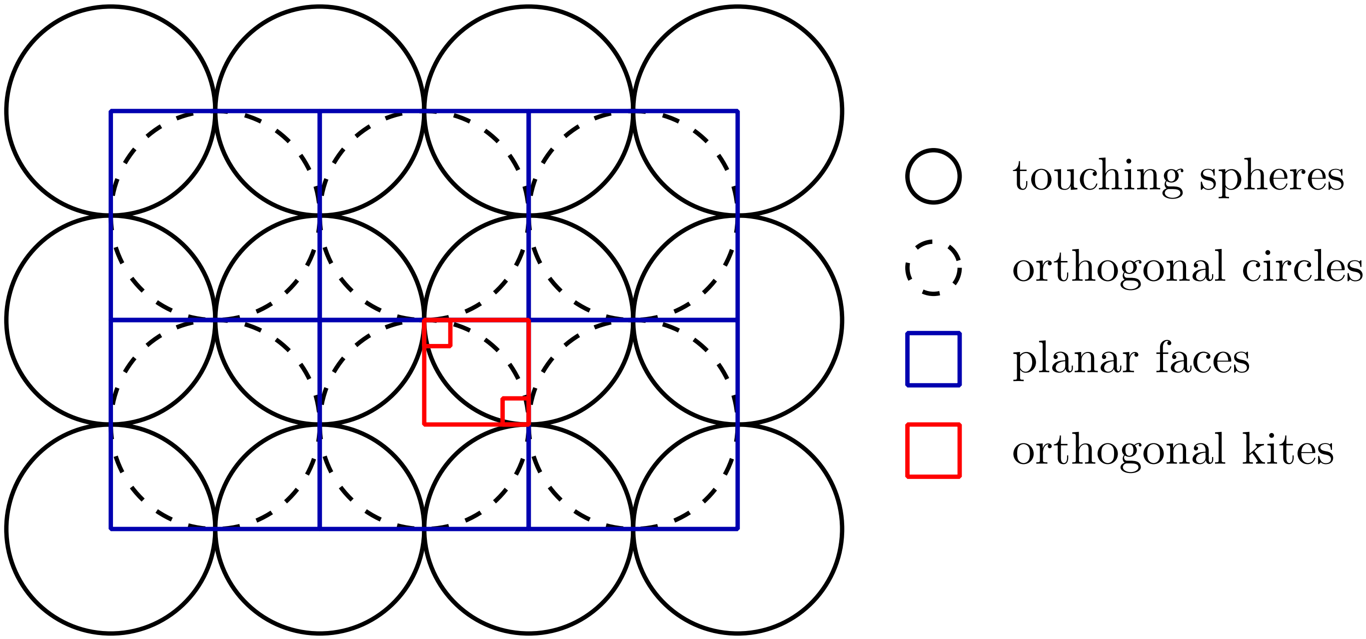


$$\frac{aa'}{bb'} = -1$$

Theorem. *Every polytopal cell decomposition of the sphere can be realized by a polyhedron with edges tangent to the sphere. This realization is unique up to projective transformations which fix the sphere.*

There is a simultaneous representation of the dual cell decomposition with orthogonanally intersecting edges.

S-isothermic discrete surfaces*

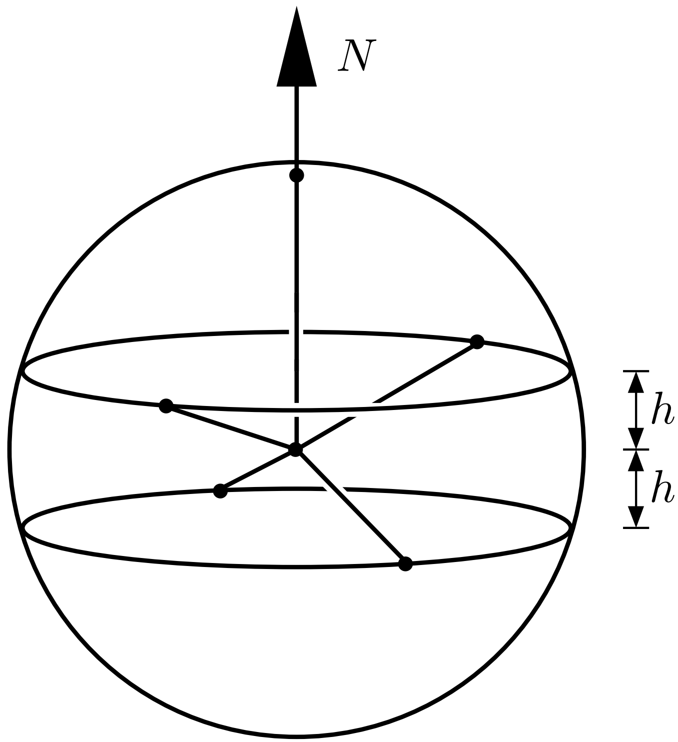


The dual of an S-isothermic surface is S-isothermic.

* [B./Pinkall '99]

Discrete minimal surfaces

Definition. *Discrete minimal* $\Longleftrightarrow \begin{cases} \text{S-isothermic,} \\ \text{extra condition at centers of spheres:} \end{cases}$



An S-isothermic surface is minimal if and only if its dual is a Koebe polyhedron.

How to construct the discrete anlogue of a continuous minimal surface

continuous minimal surface

$\Downarrow$

image of curvature lines under Gauss-map

$\Downarrow$

cell decomposition of (a branched cover of) the sphere

$\Downarrow$

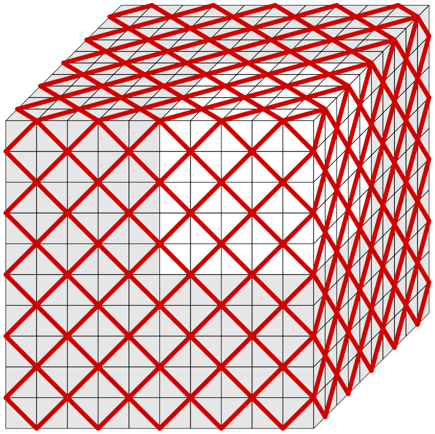
orthogonal circle pattern

$\Downarrow$

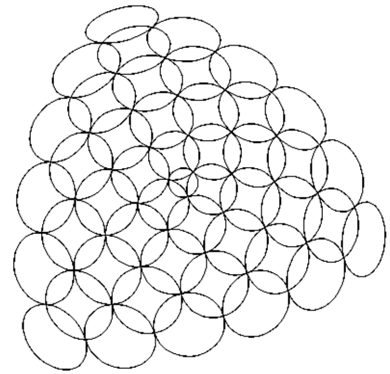
Koebe polyhedron

$\Downarrow$

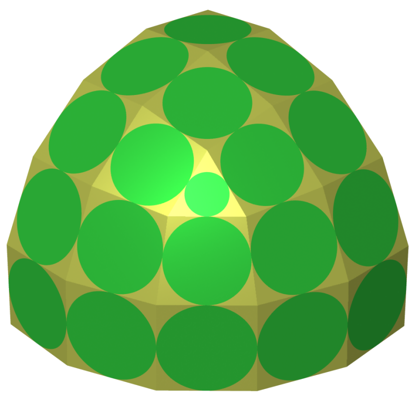
discrete minimal surface



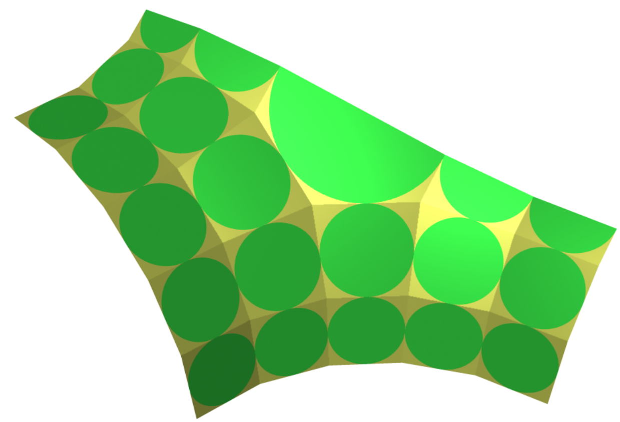
cell decomposition



circle pattern

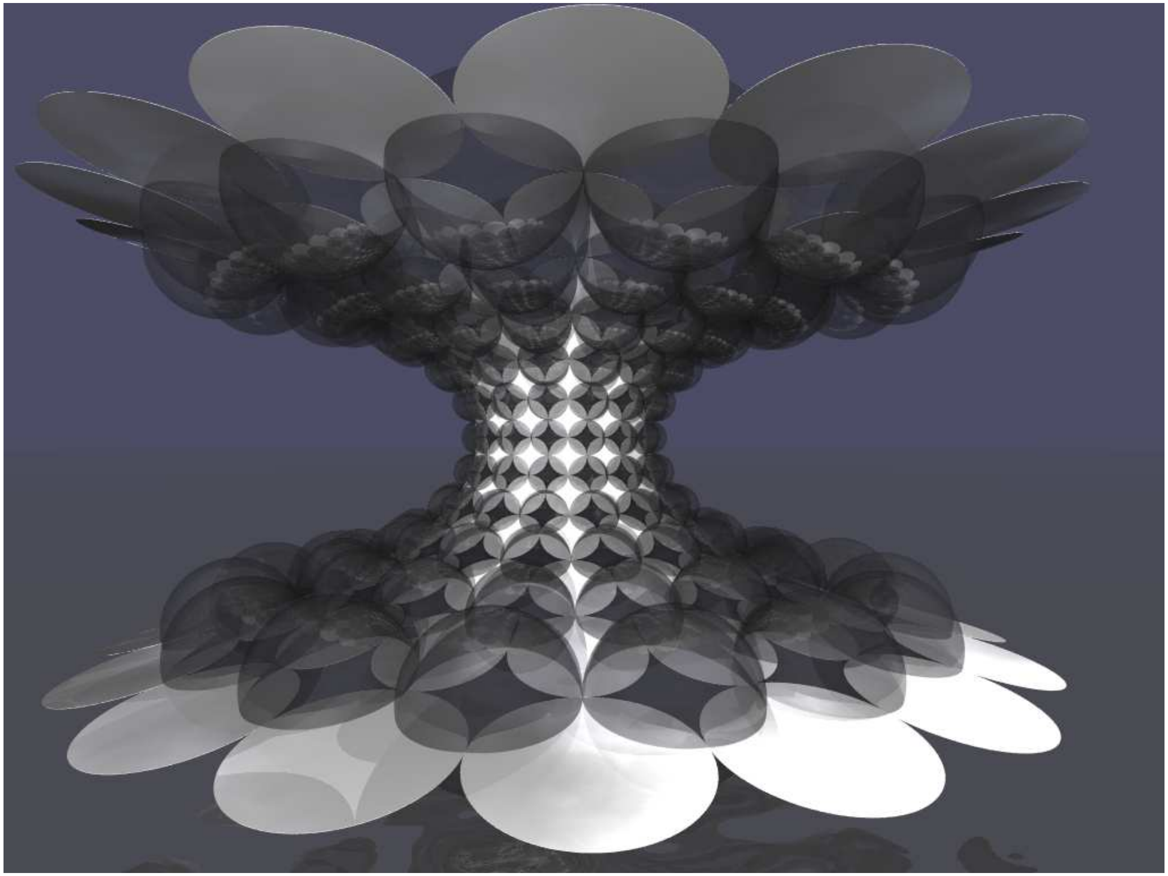


Koebe polyhedron



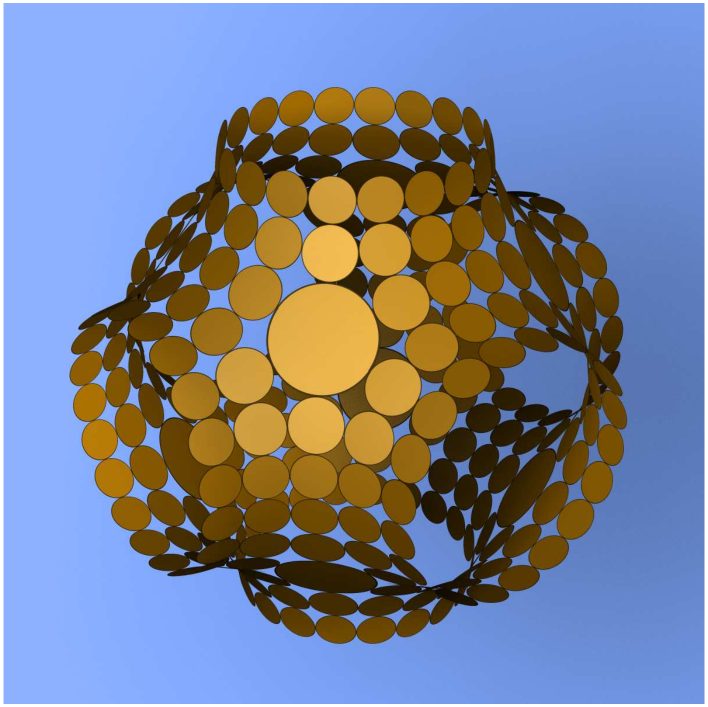
discrete minimal surface

Pictures

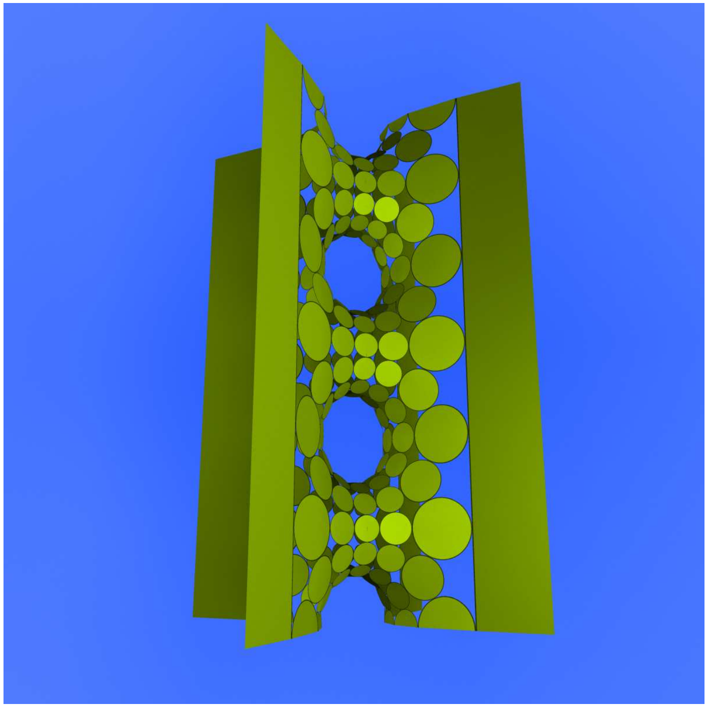


Catenoid

Pictures



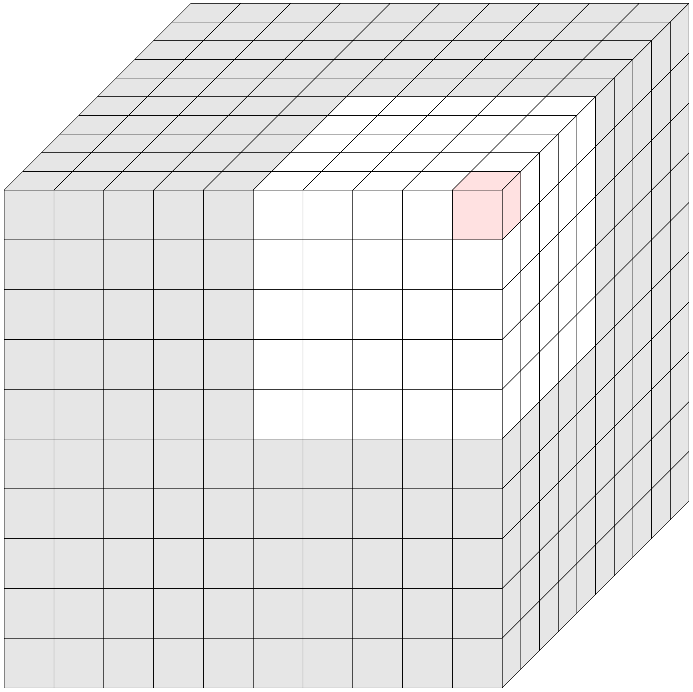
Schwarz P

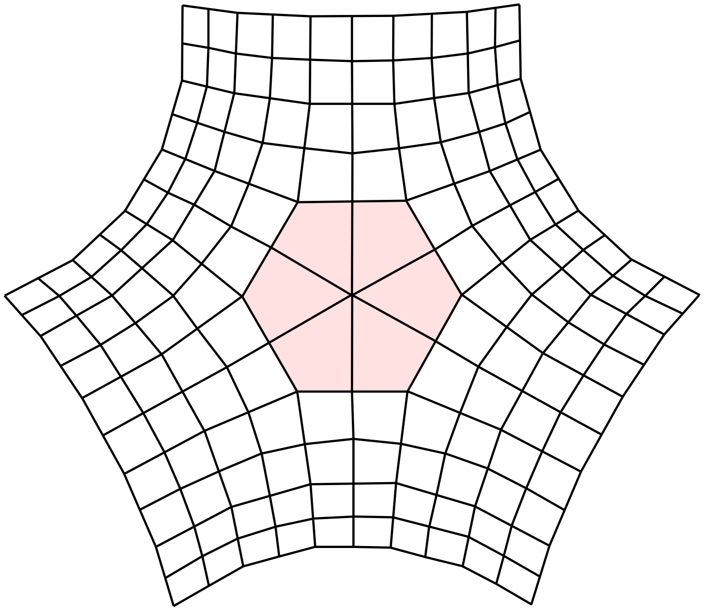


Scherk tower

The combinatorics of singularities

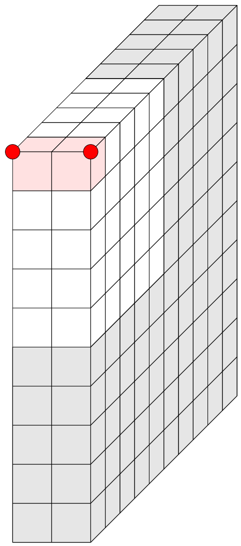
Schwarz P

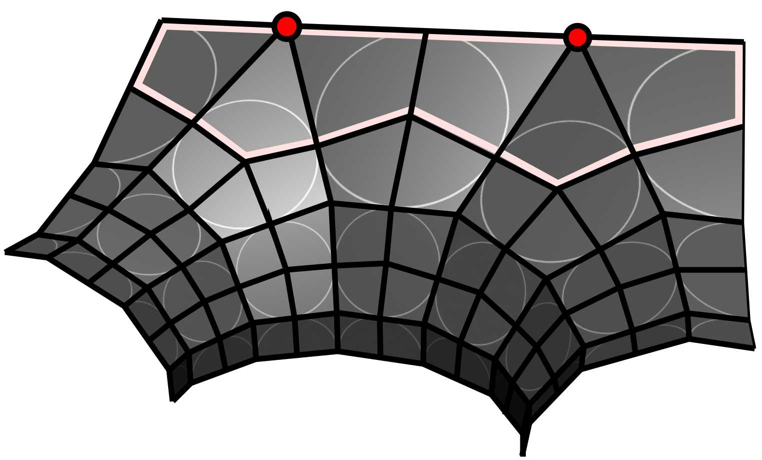


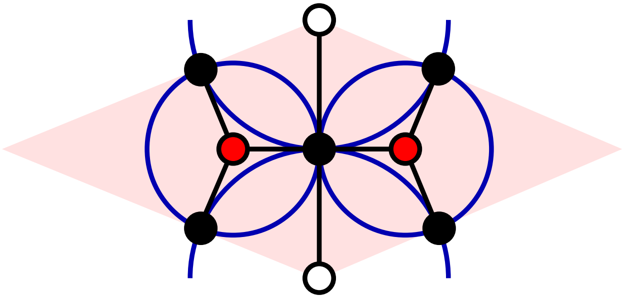


The combinatorics of singularities

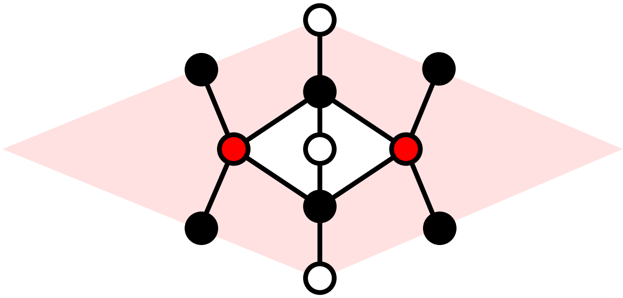
Scherk







$\longrightarrow$



Constructing orthogonal circle patterns

unknowns:

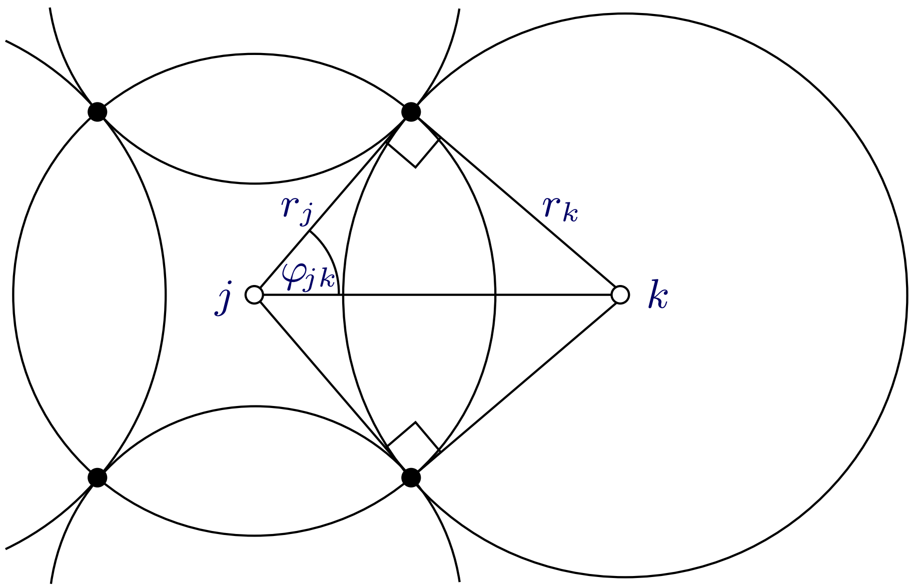
radii r

closure condition:

$$\forall j : \sum_{\text{neighbors } k} 2\varphi_{jk} = 2\pi,$$

where

$$\varphi_{jk} = \arctan \frac{r_k}{r_j}$$



How to solve the closure equations for the radii?

Constructing orthogonal circle patterns

change of variables: $r = e^{\rho}$

minimize the convex function [B./Springborn '02]

$$S(\rho) = \sum_{j \circ\!-\!\circ k} \left(\operatorname{Im} \operatorname{Li}_2\left(ie^{\rho_k - \rho_j}\right) + \operatorname{Im} \operatorname{Li}_2\left(ie^{\rho_j - \rho_k}\right) - \frac{\pi}{2}(\rho_j + \rho_k) \right) + 2\pi \sum_{\circ j} \rho_j$$

dilogarithm function: $\operatorname{Li}_2(z) = \frac{z}{1^2} + \frac{z^2}{2^2} + \frac{z^3}{3^2} + \ldots$

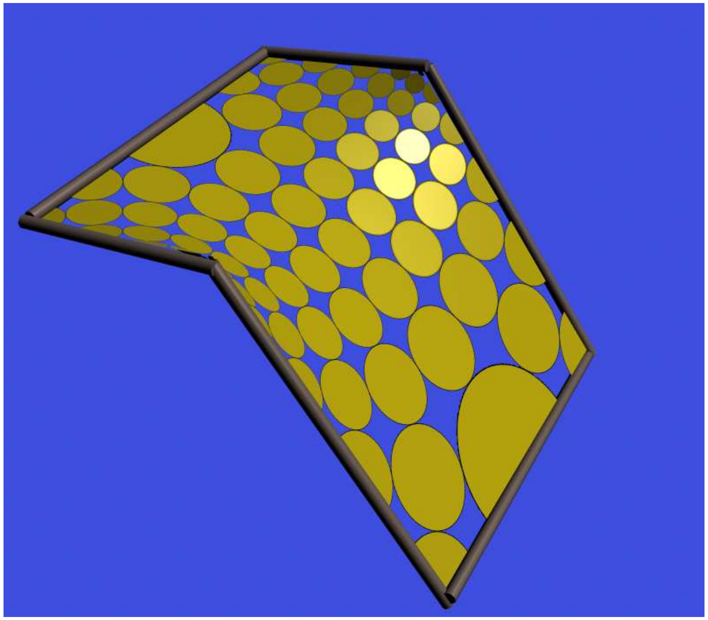
Explicit formula, no contraints, easy to compute (!)

Convexity $\Rightarrow$ uniqueness. Existence more delicate.

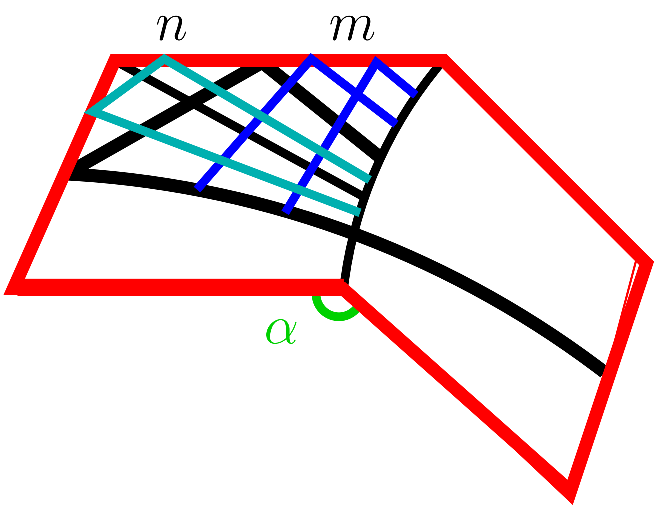
Other methods:

- Adjust, iteratively, each radius such that neighboring circles fit [Thurston]. Implemented in Stephenson's `circlepack` for packings.
- Other variational principles [Colin de Verdière '91, Brägger '92, Rivin '94, Leibon '01]

Generalization of Schwarz's CLP-surface



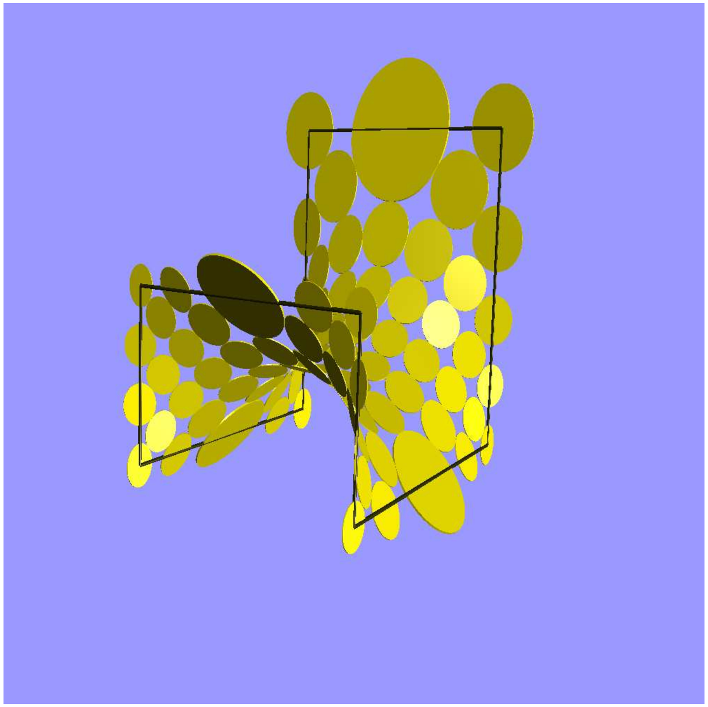
discrete minimal surface



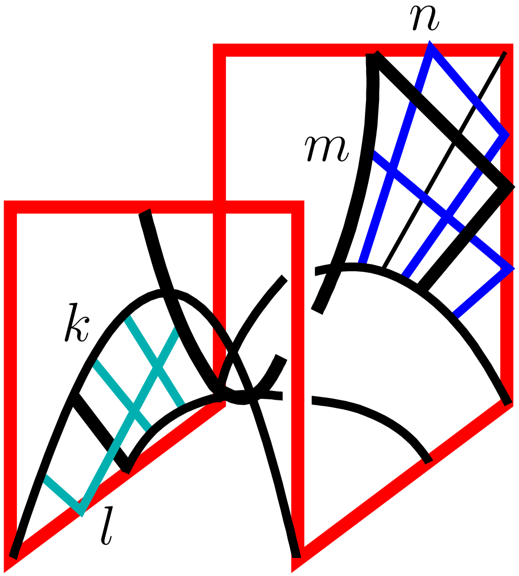
combinatorics of curvature lines

Ulrike Scheerer

Another Plateau problem



discrete minimal surface



combinatorics of curvature lines

Ulrike Scheerer

Theorems about discrete minimal surfaces

- Existence
- Uniqueness
- Convergence
- Associated family (isometry preserving the Gauss map)