Discrete differential geometry:
Surfaces made from Circles

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Papers

Discrete differential geometry

Aim: Development of discrete equivalents of the geometric notions and methods of differential geometry. The latter appears then as a limit of refinements of the discretization.

- new geometric objects in Discrete Geometry
- new methods (difference equations, geometric understanding of integrability)
- deep understanding of smooth theory (unification of surfaces and their transformations)
- solution of problems in Differential Geometry
Discrete surfaces

Triangulated (Euclidean)  Circular (Möbius)

Natural in Möbius geometry:
- curvature line parametrized surfaces
- conformally parametrized surfaces
- (triply)-orthogonal coordinate systems
Discrete orthogonal coordinate systems

**Dupin Theorem.** Coordinate surfaces of a triply orthogonal coordinate system intersect along their common curvature lines

- Circular lattices as discrete curvature line parametrization [Martin et al., Nutbourne ’86]
- Discrete orthogonal coordinate systems [B.’96]
- Cauchy problem based on Miguel’s theorem [Cieśliński, Doliwa, Santini ’97]
- Convergence with all derivatives ($C^\infty$-convergence) [B./Matthes/Suris ’03]
Discrete conformal energy

\[ W = \sum_{k} \beta_k - 2\pi \]

\( \beta_i \) are external angles between the circumscribed circles.

\( W \geq 0 \) and \( W = 0 \) iff the vertex and all its neighbors lie on a sphere.

Minimizing of \( W \) makes the surface as round as possible. Analogue of the Willmore energy

\[ W = \int (k_1 - k_2)^2 \]

- conformal
- \( W \geq 0 \) and \( W = 0 \) iff the round sphere
- \( W = \int H^2 - \int K \), on compact surfaces \( \int K \) topological invariant
- related to elastic energy
Minimizing discrete conformal energy

Triangulation: “spherical”, $W = 0$
- sphere.gif

Triangulation: “non-spherical”, $W > 0$
- nonsphere.gif

discrete Boy surface (projective plane)
- boy.gif
Minimal surface: Schwarz’ P-surface

continuous*

discrete

Isothermic surfaces

Definition. A surface in 3-space is called \textit{isothermic} if it admits conformal curvature line coordinates.

- Definition is Moebius invariant.
- Curvature lines divide the surface into infinitesimal squares.

Examples: surfaces of revolution, quadrics, constant mean curvature surfaces, minimal surfaces.
Isothermic surfaces

Definition. A polyhedral surface in 3-space is called \textit{discrete isothermic} if all faces are conformal squares, i.e., planar with cross ratio $-1$. [B./Pinkall ’96]

- Definition is Moebius invariant.
- ‘Curvature lines’ divide the surface into conformal squares.
Duality for isothermic surfaces

*continuous*

**Definition/Theorem.** If \( f : \mathbb{R}^2 \supset D \to \mathbb{R}^3 \) is an isothermic immersion, then the *dual isothermic* immersion is defined by

\[
df^* = \frac{f_x}{\|f_x\|^2} \, dx - \frac{f_y}{\|f_y\|^2} \, dy.
\]
Duality for isothermic surfaces

discrete

Proposition. Suppose \( a, b, a', b' \in \mathbb{C} \) with

\[
a + b + a' + b' = 0 \quad \text{and} \quad \frac{aa'}{bb'} = -1
\]

and let

\[
a^* = \frac{1}{a}, \quad a'^* = \frac{1}{a'}, \quad b^* = -\frac{1}{b}, \quad b'^* = -\frac{1}{b'}.
\]

Then

\[
a^* + b^* + a'^* + b'^* = 0 \quad \text{and} \quad \frac{a^*a'^*}{b^*b'^*} = -1.
\]

Can define duality for discrete isothermic surfaces if edges may be labeled ‘+’ and ‘−’ appropriately.
Minimal surfaces

- Minimal surfaces are isothermic.
- Isothermic $F$ is minimal.  $\iff F^*$ contained in a sphere.  (It’s the Gauss map.)

**A way to construct minimal surfaces:**

\[
\text{conformally parametrized sphere} \xrightarrow{\text{dualize}} \text{minimal surface}
\]

**Idea:**

\[
\text{conformally parametrized discrete sphere} \xrightarrow{\text{dualize}} \text{discrete minimal surface}
\]
Circle packings

circle packing $\leftrightarrow$ triangulation
circles $\leftrightarrow$ vertices
touching circles $\leftrightarrow$ edges

Koebe’s Theorem (1936). To every triangulation of the sphere there corresponds a circle packing. It is unique up to Moebius transformations.

“Auf diesen Schließungssatz bzw. einen damit zusammenhängenden merkwürdigen Polyedersatz beabsichtige ich in einer besonderen Note zurückzukommen, die ich der Preuß. Akademie der Wissenschaften überreichen will.”
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Orthogonal circle patterns

Red circles intersect black circles orthogonally.

**Theorem.** *To every polytopal cell decomposition of the sphere there corresponds an orthogonal circle pattern. It is unique up to Moebius transformations.*

Schramm ’92 (more general result).
Brightwell/Scheinerman ’93 (proof à la Thurston).
**Theorem.** Every polytopal cell decomposition of the sphere can be realized by a polyhedron with edges tangent to the sphere. This realization is unique up to projective transformations which fix the sphere.

There is a simultaneous representation of the dual cell decomposition with orthogonally intersecting edges.
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There is a simultaneous representation of the dual cell decomposition with orthogonally intersecting edges.
The dual of an \textit{S-isothermic} surface is \textit{S-isothermic}.

* [B./Pinkall ’99]
Discrete minimal surfaces

**Definition.** *Discrete minimal* $\iff$ \{ S-isothermic, 
extra condition at centers of spheres: \}

An *S*-isothermic surface is minimal if and only if its dual is a Koebe polyhedron.
How to construct the discrete analogue of a continuous minimal surface

continuous minimal surface
\[\Downarrow\]
image of curvature lines under Gauss-map
\[\Downarrow\]
cell decomposition of (a branched cover of) the sphere
\[\Downarrow\]
orthogonal circle pattern
\[\Downarrow\]
Koebe polyhedron
\[\Downarrow\]
discrete minimal surface
Bobenko Surfaces made from Circles

- cell decomposition
- circle pattern
- Koebe polyhedron
- discrete minimal surface
Pictures

Catenoid
Pictures

Schwarz P

Scherk tower
The combinatorics of singularities

Schwarz P
The combinatorics of singularities

\textit{Scherk}
Constructing orthogonal circle patterns

unknowns:

radii $r$

closure condition:

$$\forall j : \sum_{\text{neighbors } k} 2\varphi_{jk} = 2\pi,$$

where

$$\varphi_{jk} = \arctan \frac{r_k}{r_j}$$

How to solve the closure equations for the radii?
Constructing orthogonal circle patterns

change of variables: \( r = e^\rho \)

minimize the convex function [B./Springborn ’02]

\[
S(\rho) = \sum_{j \neq -k} \left( \text{Im} \, \text{Li}_2(ie^{\rho_k - \rho_j}) + \text{Im} \, \text{Li}_2(ie^{\rho_j - \rho_k}) - \frac{\pi}{2} (\rho_j + \rho_k) \right) + 2\pi \sum_{\rho_j}
\]

\text{dilogarithm function:} \quad \text{Li}_2(z) = \frac{z}{1^2} + \frac{z^2}{2^2} + \frac{z^3}{3^2} + \ldots

Explicit formula, no contraints, easy to compute (!)

Convexity \( \Rightarrow \) uniqueness. Existence more delicate.

Other methods:

- Adjust, iteratively, each radius such that neighboring circles fit [Thurston]. Implemented in Stephenson’s circlepack for packings.

- Other variational principles [Colin de Verdière ’91, Brägger ’92, Rivin ’94, Leibon ’01]
Generalization of Schwarz’s CLP-surface

discrete minimal surface

combinatorics of curvature lines

Ulrike Scheerer
Another Plateau problem

discrete minimal surface

combinatorics of curvature lines

Ulrike Scheerer
Theorems about discrete minimal surfaces

- Existence
- Uniqueness
- Convergence
- Associated family (isometry preserving the Gauss map)